Sample Size Requirements For Stratified Random Sampling Of Pond Water With Cost Considerations Using A Bayesian Methodology

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Abstract: Estimating average environmental pollution concentrations and its variance is a fairly straightforward task in stratified random sampling. A more challenging concept is the introduction of the cost factor into this environmental model. Traditional statistical techniques have incorporated costs from sampling within a stratum as well as stratum weights to determine the stratum size and overall required sample size. Information in the form of informative prior distributions to determine a more coherent variance in the system yield a more precise Bayesian approach to the sample size and cost calculations. This approach results in a more efficient sampling strategy in terms of cost when considering a pre specified margin of error for the sampling mean as well as the more complicated situation of correlation among the strata samples.

Keywords: Stratified; random sampling; cost; Bayesian

1. INTRODUCTION

The Traditional statistical approaches to calculating overall and stratum sample sizes in a stratified random sample are fairly straightforward. The procedure is somewhat complicated with the incorporation of cost as well as the possibility of correlation among the stratum samples. Applications of such approaches employing several monitoring strategies are well known (Thornton et al., 1982, Nelson and Ward, 1981, Reckhow and Chapra, 1983, and Gilbert, 1987). Our focus here is to consider a pond water environment in which the strata are basically depth levels. Weighting of the strata as well as the overall variance of the sample mean are the main components in our derived statistics to determine sample size within the stratum. The three situations considered are that of pre specified margin of error, pre specified fixed cost and correlation among the strata samples. Cost efficiency is seen for most situations with the introduction of Bayesian methodology (Dayal and Dickey, 1976, Bartolucci et al., 1998). The thrust of the Bayesian approach is through the derivation of the posterior estimate of the variance derived from coherent inference on a normal variance in the Behrens-Fisher context (Dayal and Dickey, 1976, Bartolucci et al., 1998). Comparisons of the traditional or classical and Bayesian methodologies are presented using summary data from determining the phosphorous concentration in a pond water sampling environment.

2. TRADITIONAL SETUP

Let N=total number of population units in the target population. \(N_h\) is the number of population units within each of the \(h\) stratum, \(h=1,\ldots,L\).

Clearly \(N=\sum_{h=1}^{L} N_h\). With reference to the sample, \(n=\sum_{h=1}^{L} n_h\). We define the weight of the stratum, \(h\),
as \( W_i = N_i/N \). The mean, \( \mu \), of the population of \( N \) units is:

\[
\mu = \left( \frac{1}{N} \right) \sum_{i=1}^{N} x_i \tag{1}
\]

where \( \mu_i \) is the mean of the \( h \) stratum and is estimated by

\[
m_h = \left( \frac{1}{n_h} \right) \sum_{i=1}^{n_h} x_{ih} \tag{2}
\]

where \( x_{ih} \) is the \( i \)th observation in stratum, \( h \). An unbiased estimate of \( \mu \) is

\[
m_h - \frac{1}{n_h} \sum_{h=1}^{H} W_h n_h m_h \tag{3}
\]

Let \( N_i/n = n_i/n \) in all strata, then

\[
m_h = \left( \frac{1}{n_h} \right) \sum_{i=1}^{n_h} x_{ih} \tag{4}
\]

We define \( \text{Var}(m_h) = \frac{1}{n_h} \sum_{i=1}^{n_h} W_i \sigma_i^2 \) where \( \sigma_i^2 \) is the variance of the \( h \) stratum. We estimate the stratum variance by

\[
s_h^2 = \frac{1}{n_h-1} \sum_{i=1}^{n_h} (x_{ih} - m_h)^2 \tag{5}
\]

It will be important to note the robustness of this sample mean variance in the Bayesian context.

### 3. COMPUTING THE OPTIMUM \( n \)

We give a brief overview of three methods to compute the optimum \( n \).

i) Pre Specified Margin of Error (PMOE)

Letting \( d = |m_{st} - \mu| \), we denote \( d \) as the pre specified margin of error, (Gilbert, 1987). The value \( d \) is such that

\[
P(|m_{st} - \mu| \geq d) = \alpha \tag{6}
\]

for small \( \alpha \). The optimum \( n \) (Cochran, 1977) is thus

\[
n = \frac{z_{1-\alpha/2}^2}{d^2} \sum_{h=1}^{H} W_h s_h^2 \tag{7}
\]

where for \( N \)

\[
n = \frac{z_{1-\alpha/2}^2}{d^2} \sum_{h=1}^{H} W_h s_h^2 \tag{8}
\]

and \( z_{1-\alpha/2} \) is the usual 100(1-\( \alpha/2 \)) critical value of the standard normal distribution. Thus the optimum \( n_h \) for the \( h \)th stratum is

\[
n_h = \frac{z_{1-\alpha/2}^2}{d^2} W_h s_h^2 \sum_{h=1}^{H} W_i s_i^2 \tag{9}
\]

ii) Pre Specified Fixed Cost

We define the overall cost of the sampling as

\[
\text{Cost} = C + \sum_{h=1}^{H} c_h \tag{10}
\]

where \( c_h \) is the cost per population unit in the \( h \)th stratum and \( c_c \) is the fixed overhead cost. This is a standard cost representation. Thus the optimum \( n \) can be derived as (Aczel, 1999),

\[
n = \frac{C - c_c \sum_{h=1}^{H} W_h s_h^2}{\sum_{h=1}^{H} W_h s_h^2 \sqrt{c_h}} \tag{11}
\]

As above the optimum \( n_h \) per stratum is

\[
n_h = \frac{C - c_c \sum_{h=1}^{H} W_h s_h^2}{\sum_{h=1}^{H} W_i s_i^2 \sqrt{c_h}} \tag{12}
\]

One can examine equation (10) in terms of its sensitivity to changes in the PMOE. Let \( W_i = n_i/n \).

Then (10) can be rewritten as

\[
n = \frac{C - c_c \sum_{h=1}^{H} n_h s_h^2}{\sum_{h=1}^{H} n_h s_h^2 \sqrt{c_h}} \tag{13}
\]

If we assume unequal PMOE, \( d_h \), for sampling within stratum then we can write \( n_h = \left(Z_{1-\alpha/2}\sqrt{n_i}/d_h \right) \). (See Cochran, 1977, Aczel, 1999). Thus equation (12) can now be examined with respect to sensitivity to changes in \( d_h \).

iii) Correlation among Depth Stratum

Let \( \rho_l \) = average correlation among all possible lags in the depth sampling environment. For example if \( L \) is the number of strata or depths and \( \rho_l \) the correlation of the \( l \)th lag, then

\[
\rho_l = \left( \frac{1}{L} \right) \sum_{i=1}^{L} P_i^l \tag{14}
\]
If \( n_h \) is the number to be sampled in each of the \( L \) strata or \( n_h = \text{stratum size} \), then

\[
n_h = z^2 \sum_{k=1}^{L} W_{sk}^2 d^{[L-k(L-1)]/L} \tag{14}
\]

4. BAYESIAN CONSIDERATIONS

Examining equations (7), (10), (12) and (14) we see that they all involve the expression for the stratum variance, \( s_h^2 \). We reevaluated these expressions adding a prior structure to the variance (Dayal and Dickey, 1977, Bartolucci et. al. 1998) and then estimating the posterior expression for the variance, normal \( \sigma^2 \). We assumed an underlying normal distribution with both mean, \( \mu \), and variance, \( \sigma^2 \) unknown. In this context we define the likelihood function for \( n \) observations:

\[
\mathcal{L}(\mu, \sigma) \propto \left(\frac{1}{\sigma}\right)^{\frac{n}{2}} \exp \left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2 + \frac{1}{2} \sigma^2 \right\} \tag{15}
\]

for \( v=n-1 \), \( nm=x_1+x_2+\ldots+x_n \), \( \nu^2=(x_1-m)^2 + (x_2-m)^2 + \ldots + (x_n-m)^2 \) and \( \propto \) denotes a proportional relationship. Consider the t-density,

\[
\phi(x; S) = S^{\frac{1}{2}} \left[ T^{1/2} \text{Beta}(v/2, 1/2) \right]^{-1} \left\{ 1 + \frac{v}{2} \left( \frac{x^2}{S} \right) \right\}^{-\frac{v+1}{2}} \tag{16}
\]

where \( \text{Beta}(a, b) = \int_0^1 z^{a-1} (1-z)^{b-1} dz \)

and \( v, S>0 \).

The prior for \( \mu \) is

\[
p(\mu) \propto \left( \frac{1}{\sigma} \right)^{\frac{n}{2}} \exp \left\{-\frac{1}{2\sigma^2} (\mu - \mu_0)^2 \right\} \tag{17}
\]

for \( \mu_0, 64 \).

The prior for \( \sigma^2 \) is

\[
p(\sigma^2) \propto \nu^2 \chi^2_t / \sigma^2, \quad \nu>0, g>0 \tag{18}
\]

Where \( \chi^2_t \) is chi square on \( \nu \) degrees of freedom.

Thus considering expressions (15), (17), and (18) the posterior variance is

\[
\sigma^2 = (us^2 + rg^2) / B \tag{19}
\]

where \( B = \nu + \tau \).

Thus substituting \( \mathcal{E}_h^2 \) for \( S_h^2 \) in (7), (10), (12), and (14) yields the Bayesian estimates of \( n \) and \( n_h \).

Thus in the following section we apply the Bayesian analysis to these expressions to demonstrate and determine the efficiency of these expressions in terms of the sample size requirements and cost of sampling.

5. EXAMPLE

We wish to estimate the average phosphorous concentration (\( \mu g/100 \text{ ml} \)) in pond water. The concentration of 100 ml aliquot from each 1 liter sample will be measured. The statistics for a classical representation of the data using the pre specified margin of error (PMOE) are given in Table 1. There are 5 depth strata to the pond in which \( N = \text{total number of 100 ml water samples in the pond} \). \( N_h \) is the number of aliquots in stratum \( h \). The weights, number samples from each strata, mean and variance of each strata are given as well all derived from our previous formulations above. We have assigned costs to each strata. For the sake of simplicity and without loss of generality we have reduced the costs to integer units. The cost for sampling stratum 1 and 2 are each 1. The costs assigned to strata 3, 4, and 5 are 2, 2, and 3 respectively - the assumption being that costs increase as the depth increases. Thus the overall cost of sampling is 74 units. Using the PMOE approach in Table 2 demonstrates the Bayesian results using empirical prior sampling information and incorporating that into the variance calculation overall. One sees that for realistic prior assignments of \( \nu, \tau \) and \( g \) in (13) that one realizes a reduction in assigned number per strata overall as well as a cost reduction. However, we did examine these results using (12) in which we varied the PMOE, \( d_h \), to determine the effect on cost using sensitivity changes and the classical and Bayesian results remained fairly equal. Table 4 summarizes the data results introducing correlation among the strata as per (14). The average correlation is in the first column. One can see that as you increase the average correlation, \( \frac{1}{\tau} \), then the required number sampled within each strata will increase, but at a slower rate in the Bayesian context.
Overall it appears that: Compared to the classical sampling analysis for the pre specified margin of error approach as well as the correlational approach, the Bayesian analysis resulted in a reduction in required samples thus lowering the cost, especially when realistic (empirical) prior hyperparameters are utilized. Also there was no serious impact on the posterior standard error of the estimates of the mean concentration. However, there were no real differences between the classical and Bayesian approaches in the pre specified fixed cost analysis. Given the current computational tools the Bayesian calculations proved to be fairly straight forward. Also given the current availability of databases, future Bayesian approaches to environmental sampling should be given serious consideration especially where costs are concerned.

REFERENCES


**Table 1.** Data for stratified random sampling to estimate samples per strata (PMOE)  
Classical Approach ($\nu = 1$, $\tau = 0$, $g = 1$) $s^2(m_u) = 0.0140$, Cost=74

<table>
<thead>
<tr>
<th>Strata</th>
<th>$N_h$</th>
<th>$W_h$</th>
<th>$n_h$</th>
<th>$m_h$</th>
<th>$s^2_{h}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>4.25M</td>
<td>0.266</td>
<td>10</td>
<td>1.67</td>
<td>0.4376</td>
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<tr>
<td>2</td>
<td>3.96M</td>
<td>0.248</td>
<td>9</td>
<td>2.83</td>
<td>0.4228</td>
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<tr>
<td>3</td>
<td>3.23M</td>
<td>0.202</td>
<td>8</td>
<td>3.59</td>
<td>0.5339</td>
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<td>2.85M</td>
<td>0.178</td>
<td>9</td>
<td>4.23</td>
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</tr>
<tr>
<td>5</td>
<td>1.70M</td>
<td>0.106</td>
<td>7</td>
<td>5.31</td>
<td>1.3920</td>
</tr>
<tr>
<td>Total</td>
<td>15.99M</td>
<td>1.000</td>
<td>43</td>
<td></td>
<td>-</td>
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</table>

**Table 2.** Bayesian Results (PMOE)

<table>
<thead>
<tr>
<th>$(\nu, \tau, g)$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$n_5$</th>
<th>Total</th>
<th>$s^2(m_u)$</th>
<th>Cost</th>
</tr>
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<td>35,1,0.5</td>
<td>9</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>41</td>
<td>0.0140</td>
<td>71</td>
</tr>
<tr>
<td>35,2,0.5</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>40</td>
<td>0.0141</td>
<td>70</td>
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<tr>
<td>20,1,1.0</td>
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<td>8</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>39</td>
<td>0.0138</td>
<td>67</td>
</tr>
<tr>
<td>40,35,0.2</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>22</td>
<td>0.0143</td>
<td>38</td>
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<tr>
<td>40,35,0.5</td>
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<td>7</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>30</td>
<td>0.0195</td>
<td>42</td>
</tr>
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</table>

**Table 3.** Pre specified fixed cost

<table>
<thead>
<tr>
<th>$C-c_0$</th>
<th>$\nu$</th>
<th>$\tau$</th>
<th>$g$</th>
<th>$n$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$n_5$</th>
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<tbody>
<tr>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>31</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>5+</td>
</tr>
<tr>
<td>50</td>
<td>40</td>
<td>35</td>
<td>0.12</td>
<td>31</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>
Table 4. Example Using the Correlation Structure, $\rho_c$.

<table>
<thead>
<tr>
<th>prior (v,t,g)</th>
<th>Classical</th>
<th>(35,1,0.5)</th>
<th>(20,1,0.1)</th>
<th>(40,35,.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_c$</td>
<td>n&lt;sub&gt;h&lt;/sub&gt; Cost</td>
<td>n&lt;sub&gt;h&lt;/sub&gt; Cost</td>
<td>n&lt;sub&gt;h&lt;/sub&gt; Cost</td>
<td>n&lt;sub&gt;h&lt;/sub&gt; Cost</td>
</tr>
<tr>
<td>0.05</td>
<td>10 90</td>
<td>10 90</td>
<td>10 90</td>
<td>05 45</td>
</tr>
<tr>
<td>0.10</td>
<td>12 108</td>
<td>12 108</td>
<td>11 108</td>
<td>06 48</td>
</tr>
<tr>
<td>0.15</td>
<td>14 126</td>
<td>13 117</td>
<td>13 117</td>
<td>07 63</td>
</tr>
<tr>
<td>0.25</td>
<td>17 153</td>
<td>16 144</td>
<td>16 144</td>
<td>09 81</td>
</tr>
<tr>
<td>0.35</td>
<td>21 189</td>
<td>20 180</td>
<td>19 171</td>
<td>11 99</td>
</tr>
<tr>
<td>0.45</td>
<td>24 216</td>
<td>23 207</td>
<td>22 176</td>
<td>12 108</td>
</tr>
<tr>
<td>0.55</td>
<td>28 252</td>
<td>26 234</td>
<td>25 225</td>
<td>14 126</td>
</tr>
</tbody>
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