Assessing Rainfall-Runoff Records for Time Invariance using Transfer Function Models with Time-varying Parameters

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Abstract: Most catchment-scale rainfall-runoff models represent as time-invariant such features of the hydrograph as lag to peak, volumetric throughput and recession time constant. While this assumption has been shown to suit many catchment-modelling purposes, it obscures any variation in response characteristics over the range of conditions in which the model is calibrated. The assumption of time invariance is tested by use of linear, transfer-function models with *time-varying* parameters. The approach is illustrated on seven years' daily rainfall and streamflow records from a 10.5 km^2 forested catchment in Virginia. Models are fitted by recursive minimum-covariance estimation with optimal smoothing, treating various subsets of the model parameters as time-varying, modelled as random walks. The extent of time variation is specified by variances for the parameter increments. These are chosen by reference to the root-mean-squared value of the residuals, the ratio of mean-square values of the one-step-prediction errors and residuals, and the credibility of the parameter variations. Care is found to be necessary because variations in gain and dominant time constant interact through the transfer-function denominator coefficients. The role of a time-varying output-offset term in the model is examined. Significant time variation is found in volumetric throughput and recession time constant, as a consequence of non-linearity and unmodelled or incompletely modelled phenomena such as evapotranspiration, varying soil moisture and snowmelt. The broad conclusion is that a substantial reduction in residuals and improvement in short-term prediction performance is obtainable by representing the catchment behaviour as linear but time-varying. The results indicate the extent of variation of linear-model and hydrological parameters.

Keywords: *Transfer functions; rainfall-runoff; time-varying*

1. INTRODUCTION

It is well established (Jakeman and Hornberger, 1993; Ye et al., 1997) that catchment-scale daily rainfall-runoff relations can be adequately modelled for many purposes by models with linear, low-order, time-invariant dynamics from effective rainfall to flow; a non-linear module accounts for evapotranspiration and soil moisture (Evans and Jakeman, 1997). The assumed constancy of such hydrograph features as time to peak, volumetric throughput and recession time constant is usually satisfactory when the model has to represent average behaviour over periods comparable to the calibration period.

It is possible to test how far the assumption of time invariance is met by shorter-term and large-event behaviour by fitting *time-varying*, linear, low-order models then examining the variation of their parameters and the changes in model performance. If a linear, lumped model is to approximate closely a non-linear, distributed dynamical system such as a catchment, its parameter values must generally depend on state and input. The parameter variation of a time-varying, linear rainfall-runoff model should therefore be interpretable by reference to soil moisture (storage state), rainfall and any other significant input such as snowmelt.

The idea of identifying linear, time-varying models for environmental applications is not new (Young et al., 1991). Computationally cheap recursive algorithms for estimating the parameters of such models have long been available (Norton, 1975; Young, 1984). Even so, the approach remains underexploited. This paper will illustrate its ability to yield insight into circumstances where catchment behaviour is not well modelled by time-invariant transfer-function models. In such circumstances, examination of the nature of the time variation can suggest how the model should be modified or replaced. Some potential pitfalls in estimating linear, time-varying models are also pointed out.

2. PARAMETER ESTIMATION

2.1. Model structure

The discrete-time model is of the ARMAX form

$$
\begin{aligned}
y_t &= -a_1(y_{t-1} - d_{t-1}) - \ldots \\
&\quad - a_{n_a}(y_{t-n_a} - y_{t-n_a}) + b_1 u_{t-1} + \ldots \\
&\quad + b_{n_b} u_{t-n_b} + c_1 e_{t-1} + \ldots + c_{n_c} e_{t-n_c} \qquad (1) \\
&\quad + d_t + e_t \\
&\equiv A'(q^{-1})(y_t - d_t) + B(q^{-1})u_t \\
&\qquad + C(q^{-1})e_t + d_t + e_t
\end{aligned}
$$

where t denotes time, y is streamflow, u is rainfall, $A'(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$ are polynomials in the one-sample-delay operator (so, for instance, $q^{-1}u_t \equiv u_{t-1}$). In the "noise" model, intended to model structured output behaviour not explained by the input, $C(q^{-1})$ is monic, $\{e\}$ is a white, zero-mean noise-generating sequence. The constant or slowly varying offset d accounts for flow components too slow and/or uncorrelated with rainfall to be covered by the rest of the model. The parameters to be estimated are d and the coefficients in A', B and C. Defining $A(q^{-1})$ as $1+A'(q^{-1})$, the model can be rewritten in transfer-function (output-error) form as

$$
y_t = \frac{B(q^{-1})}{A(q^{-1})} u_t + \frac{C(q^{-1})}{A(q^{-1})} e_t + d_t \qquad (2)
$$

The poles of the model are the zeros of $A(q^{-1})$. Two quantities derived from the model are particularly informative: the steady-state gain $g_{ss}=B(1)/A(1)$ (which can be interpreted as a possible time-varying area under the unit hydrograph) and the dominant (recession) time constant t_d, which is related to the dominant pole p_d of this discrete-time model by $p_d=\exp(-T/t_d)$, where T is the sampling interval of u and y. Hence $t_d = -T/\ln|p_d|$ (with the modulus signs providing for the pole turning out to be complex, although it is, of course, real if the model is hydrologically credible).

2.2. Estimation algorithm

The parameter-estimation algorithm is derived elsewhere (Norton, 1975, 1976, 1986) and a detailed understanding of it is unnecessary here, but will be outlined to put it in context and show its assumptions. It is recursive, i.e. it steps through the input-output records, updating estimates of all the unknown parameters as it goes. Its essential components are

- optimal smoothing (retrospective, minimum-covariance, linear, unbiased state estimation) applied to model parameters represented as the elements of a state vector **x**, in which the variables are represented either as constant or as executing random walks;
- an observation equation consisting of (1) with the lagged input and output-minus-offset values on the right-hand side treated as known, forming the elements of a "regressor" vector **h**;
- a structured noise model to avoid bias; and
- approximation of the unknown e_{t-i}, $1 \le i \le n_c$, by residuals $\hat{e}_{t-i} = y_{t-i} - \mathbf{h}_{t-i}^T \hat{\mathbf{x}}_{t-i}$ from earlier recursion steps, as in extended least squares, and approximation of the terms d_{t-i}, $1 \le i \le n_a$ attached to the earlier output samples by their estimates from earlier steps.

The parameters have specified mean-squared (m.s.) values for their increments (white, zero-mean sequences driving the random walks). In contrast to most recursive algorithms, which pass only once through the records, optimal smoothing performs a reverse pass through the records after an initial forward pass, so as to incorporate the information about the parameters present in the later output observations as well as the earlier ones. The forward pass is

$$
\left.
\begin{aligned}
&\mathbf{P}'_t = \mathbf{P}_{t-1} + \mathbf{Q}_{t-1}; \ \mathbf{g}_t = \mathbf{P}'_t \mathbf{h}_t \\
&m_t = 1 + \mathbf{g}_t^T \mathbf{h}_t; \ \mathbf{P}_t = \mathbf{P}'_t - \mathbf{g}_t \mathbf{g}_t^T / m_t \\
&?_t = y_t - \mathbf{h}_t^T \hat{\mathbf{x}}_{t-1}; \hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t-1} + \mathbf{g}_t ?_t / m_t
\end{aligned}
\right\} \quad (3)
$$

for $t=1,2,\ldots,N$, followed by the reverse pass

$$
\left.
\begin{aligned}
&\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t+1} - (\mathbf{g}_{t+1}^T \boldsymbol{\mu}_{t+1} + n_{t+1}) / m_{t+1} \\
&\hat{\mathbf{x}}_{t|N} = \hat{\mathbf{x}}_t - \mathbf{P}_t \boldsymbol{\mu}_t
\end{aligned}
\right\} \quad (4)
$$

back to t=0. Here $\mathbf{P}_t$ is the covariance of $\hat{\mathbf{x}}_t$, normalised by the m.s. value of $e_t = y_t - \mathbf{h}_t^T\mathbf{x}_t$ .Its principal diagonal elements indicate the variances of the estimates of the individual parameters. The "state" and "observation" vectors are, in detail,

$$\mathbf{x} \equiv \left[a'_1 \ldots a'_{na} \quad b_1 \ldots b_{nb} \quad c_1 \ldots c_{nc} \quad d\right]^T,$$

$$\mathbf{h}_t \equiv [\hat{d}_{t-1} - y_{t-1} \ \ldots \ \hat{d}_{t-n_a} - y_{t-n_a}$$

$$u_{t-1} \ldots u_{t-n_b} \quad \hat{e}_{t-1} \ldots \hat{e}_{t-n_c} \quad 1]^T \qquad (5)$$

and $\hat{\mathbf{x}}_{t|N}$, $0 \le t \le N$, signifies the estimates based on the entire N-point records. The user must provide initial parameter estimates $\hat{\mathbf{x}}_0$ and their error covariance $\mathbf{P}_0$. By default, $\hat{\mathbf{x}}_0 = \mathbf{0}$ and $\mathbf{P}_0$ is diagonal with large non-zero elements, say $10^6\mathbf{I}$ if the unknown parameters are thought to be of the order of unity. A diagonal matrix $\mathbf{Q}_{t-1}$ specifies the m.s. values of the parameter changes from time t-1 to time t (zero for constant parameters); for simplicity, the same $\mathbf{Q}$ is used throughout any one run, with single values for all coefficients in A' and all coefficients in B.

3. TESTS

3.1. Records

The records are daily rainfall and streamflow from October 1, 1992 to August 16, 1999 for the 10.5 km^2 catchment of the Staunton River on the eastern flank of the Blue Ridge in Virginia, USA. An accompanying temperature record allows calculation of effective rainfall (Jakeman and Hornberger, 1993); models using both raw and effective rainfall are examined below.

A companion paper (Chanat and Norton, 2003) discusses the peculiarities of the records and catchment. The aims here are broader: to see, in a typical example, how time-varying A, B and d deal with various aspects of the response, how model performance and derived quantities such as dominant time constant and steady-state gain are influenced by time variation of the parameters, and how anomalies and causes can be identified.

3.2. Runs

About 30 runs were performed with raw rainfall as input and a similar number with effective rainfall. The runs have differing extents of time variation of the parameters, as prescribed by the mean-square increments in their random walk models, i.e. the diagonal elements of $\mathbf{Q}$. The main indicators used to compare runs are the r.m.s. value r_e of the residuals, the ratio r =*(m.s. 1-step prediction error)/(m.s. residual)* and the credibility of the variations in the parameters, as shown by the steady-state gain g_{ss} and dominant time constant t_d. Monitoring of r is crucial to avoid excessive parameter variation due to too-high $\mathbf{Q}$: as $\mathbf{Q}$ is raised, r_e continues to fall because the model is permitted to follow short-term output behaviour more rapidly, but ultimately the parameter variation is influenced too much by noise not reflecting the catchment dynamics. Deterioration in prediction performance is then detected by a rise in r from very near unity, where it remains so long as the model follows only genuine response changes.

The results reported below were obtained with n_a=n_b=n_c=3 unless otherwise stated. This model structure is conservative (not strictly parsimonious); the transfer-function numerator and denominator orders were chosen to be slightly higher than the order, 1 or 2, typically found necessary to model long-term average behaviour, to provide flexibility in tracking possibly complicated changes in dynamics and possibly varying transport delays. The noise order is a compromise between flexibility and parsimony. Section 4.5 discusses alternative model structures.

4. RESULTS AND COMMENTS

4.1. Varying transfer function and offset

First we compare the model representing all parameters as time-invariant with the model with time-varying A, B and d, for specified m.s. changes q_A=10^{-5}, q_B=10^{-6} and q_d=10^{-3} (roughly tuned to give credible variation and r near 1). The time-varying model has r_e 48.7% and 49.3% lower for raw and effective rainfall inputs respectively, with r =1.023, 1.032 indicating negligibly worse prediction. [Inclusion of d in the time-invariant model reduces r_e by less than 1%]. The large improvement due to time variation shows that the time-invariant model misses significant short-term catchment behaviour.

Next each of A, B and d in turn is allowed to vary, with q_A ,q_B ,q_d as above in the first instance.

4.2. Varying transfer function denominator

Varying A alone reduces r_e by 4.7% and 7.8% for raw and effective rainfall, with r =1.002 for both. Increased q_A=10^{-4} yields reductions of 7.7% and 12.1% at acceptable r =1.02,1.01, but the time constant t_d shows strong correlation with flow (Figure 1, middle trace).

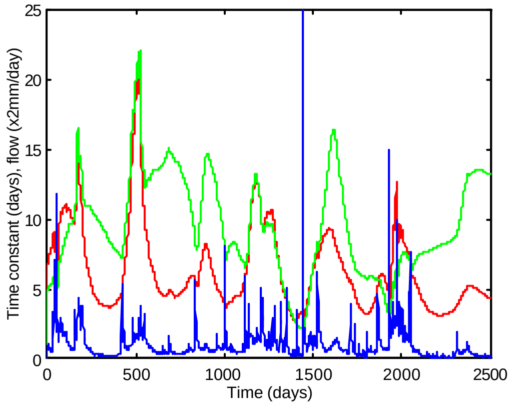

Figure 1. Dominant time constant for 3rd-order transfer-function raw rainfall-runoff model with only A varying (middle), both A and B varying (top). Bottom trace is flow.

This is hard to justify, as the response is expected to become slower, not quicker, in recessions. If both A and B vary, with q_A=10^{-4} and q_B=10^{-5}, the behaviour of the time constant during recessions is markedly more credible (Figure 1, top). It now has sharp minima in large flow events and is higher except at a few peaks. The explanation is that when A alone varies, it has to rise in periods of low steady-state gain (recessions), as B cannot decrease. Since $A(q^{-1})$ @ $1\text{-}p_dq^{-1}$ for a dominant, positive, real pole p_d, a rise in A implies a fall in p_d, shortening the time constant. Allowing B to vary provides a direct way to accommodate gain variations, leaving A to track the time constant.

Also in Figure 1, the occurrence of peaks in time constant, some sharp, at days 177, 512-513, 525, 907 and 1183, close to minor flow peaks, needs explanation. The peaks are due to rain-induced snowmelt (Chanat and Norton, 2003), which causes relatively slow rises compared with those due to saturating rainfall.

4.3. Varying transfer function numerator

Varying B alone with q_B=10^{-6} reduces r_e by 32.3% and 38.0% at r =1.017, 1.025 for raw and effective rainfall, while q_B=10^{-5} reduces r_e by 46.2% and 50.1% but with a sharp rise in r to 1.112, 1.117. However, if A is allowed to vary with q_A=10^{-4} at q_B=10^{-6}, r_e is reduced further by 13.6% and 20.6% for raw and effective rainfall inputs, with r rising only to 1.034 and 1.039. Figure 2 shows the time constant t_d for both rainfall records. With effective rainfall as input, rises in time constant during recessions are more apparent, and the height of the peak produced by snowmelt near time 512 is reduced.

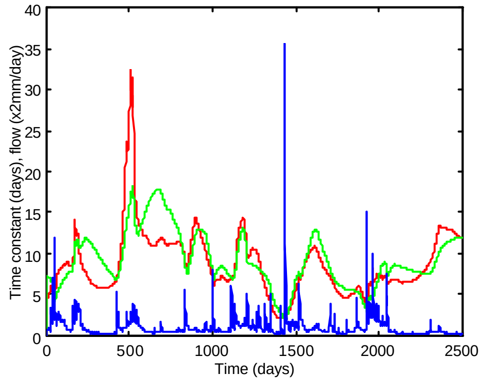

Figure 2. Dominant time constant with A and B varying and no offset; raw rainfall (top trace near t=512), effective rainfall (middle trace near t=512) as input. Bottom trace is flow.

Figure 3 shows that use of effective rainfall removes much of the need for variation in gain g_{ss} when using raw rainfall, as intended. Even so, enough variation remains to suggest that more model augmentation is desirable.

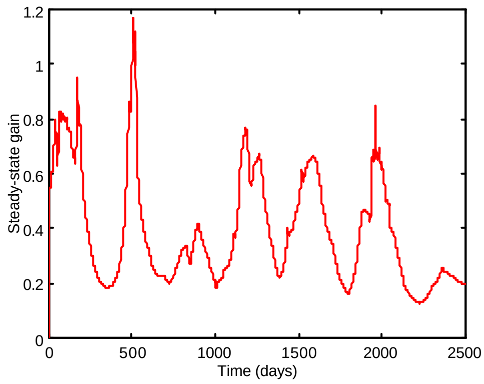

Figure 3. Steady-state gain with both A and B varying and no offset,; raw rainfall (upper), effective rainfall (lower) as input.

4.4. Inclusion of offset in model

Next the effect of adding an offset is examined. With raw rainfall as input, setting q_d=0.0025 gives the offset compared with flow in Figure 4.

The offset acts as a credible baseflow estimate, reaching zero near the ends of long recessions, but for one anomaly: an excursion below zero following the huge event at day 1440. Here interestingly the gain is overestimated, and this persists for some time because of the constraints on parameter variation imposed by **Q**.

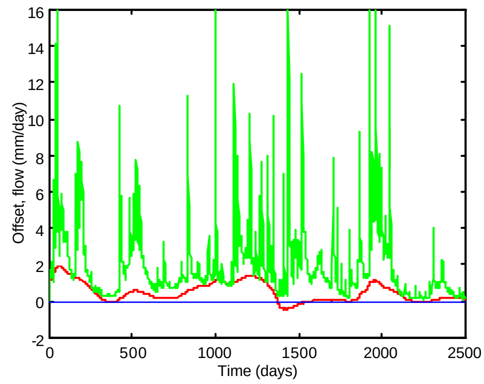

Figure 4. Offset with A, B and d varying. Raw rainfall as input.

The addition of the offset reduces r_e only by 2.0%, at almost exactly the same r, but, as Figure 5 shows, takes up enough of the slow variation in flow to reduce the variation of g_{ss} by about a third.

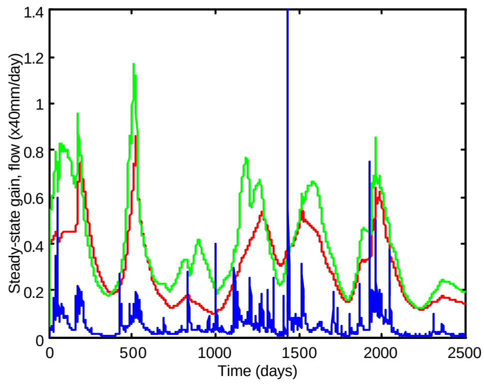

Figure 5. Steady-state gain with *A* and *B* varying: no offset (top), q_d = 0.0025 (middle), with raw rainfall as input. Bottom trace is flow.

The effect on the dominant time constant, Figure 6, is to flatten all but the largest peaks and to bring the estimates using raw and effective rainfall considerably closer together.

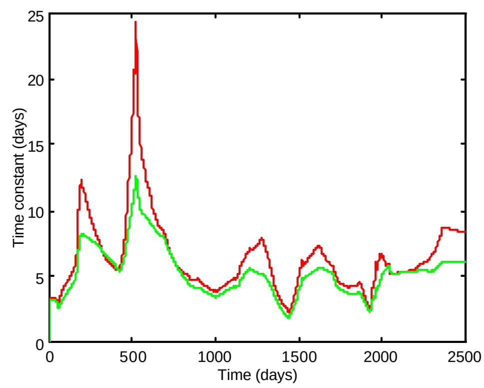

Figure 6. Dominant time constant, with *A*, *B* and *d* varying. Raw (upper) and effective (lower) rainfall as input.

With effective rainfall as input, the offset reduces r_e by 2.9% and leaves r at 1.039. In contrast to the result when raw rainfall is used, the offset is never close to zero and varies much less (like the gain), as would be expected if the effective rainfall computation took good account of evapotranspiration loss and soil moisture variation. However, the offset exceeds the flow during every recession (Figure 7), which is possible evidence that the loss is underestimated by the transfer function section of the model during recessions. This is supported by the fact that, even with effective rainfall as input, the gain g_{ss} falls during recessions, indicating underestimation of the loss (for the calculation of which the parameters were not carefully tuned).

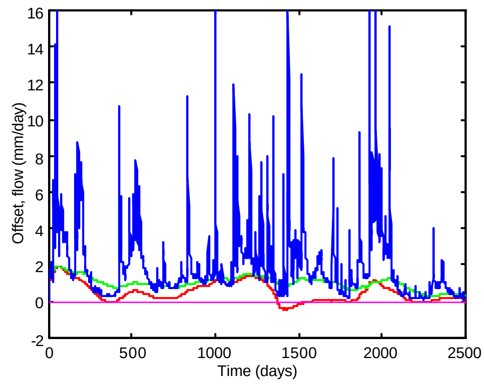

Figure 7. Offset with *A*, *B* and *d* varying: effective (upper) and raw (lower) rainfall as input.

The importance of the non-linear evapotranspiration loss computation, if a constant-parameter, linear model is to be used for the rainfall-runoff relation, is demonstrated by Figure 8. It shows that use of effective rainfall achieves a large reduction in seasonal variation of the steady-state gain of the catchment.

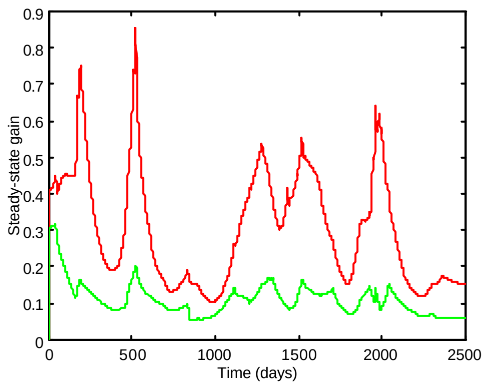

Figure 8. Steady-state gain, with *A*, *B* and *d* varying. Raw (upper) and effective (lower) rainfall as input.

The remaining variation, by about a factor of two, stems from unmodelled influences of changing transport delay (visible on close inspection of the records), non-linearities in the flow processes during extreme events, inhomogeneous gauging errors (Chanat and Norton, 2003) and snowmelt, as well as the possibly suboptimal tuning of the loss-module parameters noted above.

4.5. Other model orders

The results so far are for models with n_a=n_b=n_c=3, set high enough to minimise the risk of missing significant behaviour. A brief investigation of simpler models with q_A ,q_B ,q_d as in Section 4.4 shows that n_a=n_b=2 gives essentially the same results, but reducing n_A to 1 while keeping n_b=n_c=3 (making the transfer function improper but not unphysical) raises r_e only by 4.5% and 5.5% for raw and effective rainfall inputs, and leaves r almost unchanged at 1.042 and 1.042. The reduced freedom in A has the effect of transferring some of the observed flow from transfer function output to offset, without greatly affecting the smoothness or shape of either. One might infer that overparameterised models risk overestimating gain and underestimating baseflow.

5. CONCLUSIONS

The effects of representing selected parts of a linear rainfall-runoff model as time-varying have been investigated. Substantial reductions in r.m.s. residual are obtainable with negligible loss of one-step predictive power. Interaction between gain and dominant time constant through the coefficients of the transfer function denominator is found problematical unless adequate ability to track gain changes is conferred by time-varying numerator coefficients. Inclusion of a time-varying offset term, acting as a baseflow estimator, has been found effective when raw, but not effective, rainfall is the input. The behaviour of the offset and the steady-state gain throws some light on the extent to which the effective-rainfall calculation has succeeded.

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