Statistical Properties of STAR-GARCH Models: An Empirical Evaluation

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Abstract: Non-linear time series models, especially regime-switching models, have become increasingly popular in the economics, finance and financial econometrics literature. However, much of the research has concentrated on the empirical applications of various models, with little theoretical or statistical analysis associated with the structure or asymptotic theory. Some structural and statistical properties have recently been established for the Smooth Transition Autoregressive (STAR) - Generalised Autoregressive Conditional Heteroscedasticity (GARCH), or STAR-GARCH, model, including the necessary and sufficient conditions for the existence of moments, and the sufficient condition for consistency and asymptotic normality of the (Quasi)-Maximum Likelihood Estimator ((Q)MLE). While these moment conditions are straightforward to verify in practice, they may not be satisfied for the GARCH model if the underlying long run persistence is close to unity. A less restrictive condition for consistency and asymptotic normality may alleviate this problem. The paper evaluates a weak sufficient, or log-moment, condition for consistency and asymptotic normality of the QMLE for the STAR-GARCH model. This condition can easily be extended to any non-linear conditional mean model with GARCH errors, subject to appropriate regularity conditions. Although the log-moment condition cannot be verified as easily as the second and fourth moment conditions, it allows the long run persistence of the GARCH process to exceed one. The sufficient conditions for consistency and asymptotic normality are verified empirically using S&P 500 returns, US 3-month Treasury Bill rates, and the exchange rate between Australia and the USA.

Keywords: STAR, GARCH, STAR-GARCH, moment conditions, log-moment, non-linear time series, outliers, extreme observations.

1 Introduction

Engle's (1982) Autoregressive Conditional Heteroscedasticity (ARCH) model and Bollerslev's (1986) Generalised ARCH (GARCH) model are the most popular models for capturing time-varying symmetric volatility in financial and economic time series data. Despite their popularity, the structural and statistical properties of these models were not fully established until recently. However, most of the theoretical results on GARCH models have assumed a constant or linear conditional mean, and it has not yet been established whether those results would also hold if the conditional mean were non-linear.

Ling and McAleer (2003) proposed a multivariate ARMA - GARCH model, and established its structural and statistical properties. Jeantheau (1998) established consistency results of estimators for multivariate GARCH models. His proofs of consistency did not assume a particular functional form for the conditional mean, but assumed a log-moment condition and some regularity conditions for purposes of identification. Chan and McAleer (2002) established the structural and statistical properties for the GARCH components in the Smooth Transition Autoregressive - GARCH (STAR-GARCH) model. They showed that the results in Ling (1999) and Ling and McAleer (2002a, b, 2003) also applied in the case of STAR-GARCH, including the necessary and sufficient conditions for the existence of moments, and a sufficient condition for consistency and asymptotic normality of the (Quasi-) Maximum Likelihood Estimator ((Q)MLE).

This paper extends the results of Elie and Jeantheau (1995), Jeantheau (1998), Boussama (2000) and Chan and McAleer (2002), and shows that a weaker log-moment condition derived by Bougerol and Picard (1992) is sufficient to ensure consistency and asymptotic normality of the (Q)MLE for the GARCH component in a STAR-GARCH model. Moreover, the results of this paper can easily be extended to a wide class of non-linear time series models with GARCH errors, subject to appropriate regularity conditions.

Finally, the Logistic STAR-GARCH (LSTAR - GARCH) and Exponential STAR-GARCH (ESTAR - GARCH) models are estimated using S&P 500 Composite Returns, US 3-month Treasury Bill returns, and the exchange rate between the USA and Australia. The rolling empirical log-moment and second and fourth moment conditions, and their sensitivity to outliers and extreme observations, are also examined in detail.

The plan of the paper is as follows: Section 2 provides a brief review of the GARCH and STAR-GARCH models, with a particular emphasis on their theoretical developments. A new theoretical result regarding the statistical properties of the QMLE for STAR-GARCH is also established. The empirical results are presented in Section 3, and Section 4 gives some concluding remarks.

2 The Models

This section discusses some of the most recent theoretical results on the GARCH, STAR and STAR-GARCH models. Definitions, regularity conditions and sufficient conditions for the existence of moments, stationarity and ergodicity, and sufficient conditions for consistency and asymptotic normality of the QMLE for these models, will be discussed in detail. A new and weaker sufficient condition for consistency and asymptotic normality for the QMLE of the STAR-GARCH model will also be presented.

Let (Ω, A, P) be a probability space, $\{y_t, t \in \mathbb{Z}\}$ an $\mathbb{R}$-valued process, and $\theta = (\phi, \omega, \alpha, \beta)'$ a parameter in $\Theta \in \mathbb{R}^{k+1}$, so that $\phi = (\phi_1, \phi_2, ..., \phi_r)'$, $\alpha = (\alpha_1, ..., \alpha_p)'$, $\beta = (\beta_1, ..., \beta_q)'$, $r + p + q = k$, and θ_0 denote the true parameter vector. Define y_t as a discrete-time stochastic process with generalised conditional heteroscedastic errors if, $\forall\ t \in \mathbb{Z}$,

$$y_t = f(x_t; \phi) + \varepsilon_t \tag{2.1}$$

$$\varepsilon_t = \eta_t \sqrt{h_t}, \qquad \eta_t \sim \mathrm{iid}(0,1) \tag{2.2}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}, \tag{2.3}$$

where $x_t = (y_{t-1}, y_{t-2}, ..., \varepsilon_{t-1}, \varepsilon_{t-2}, ..., z_t)'$ and z_t is a $1 \times g$ vector of exogenous. Moreover, it is assumed that $\alpha_i > 0$ for all $i = 1, ..., p$ and $\beta_i > 0$ for all $i = 1, ..., q$ to ensure the positivity of h_t. When $q = 0$, equation (2.3) reduces to Engle's (1982) ARCH(p) process.

Define the likelihood function to be

$$l(\theta) = -\frac{1}{2T} \sum_{t=1}^{T} \big(\log h_t + \frac{\varepsilon_t^2}{h_t} \big). \tag{2.4}$$

The maximum likelihood estimator (MLE) for the model defined in equations (2.1) - (2.3) is the solution to the following maximisation problem:

$$\hat{\theta} = \mathrm{argmax}_{\theta \in \Theta} l(\theta), \tag{2.5}$$

if η_t is normally distributed. Otherwise, $\hat{\theta}$ is defined as the Quasi-MLE (QMLE).

Chan and Tong (1986) and Terašvirta (1994) extended the Threshold Autoregressive (TAR) model of Tong (1978) and Tong and Lim (1980) to allow for smooth transition behaviour, that is,

$$y_t = \sum_{i=1}^{m} \phi_i' x_t (G_{i-1}(s_t; \gamma_{i-1}, c_{i-1}) - G_i(s_t; \gamma_i, c_i)) + \varepsilon_t,$$

$$\varepsilon_t \sim iid(0, \sigma^2), \tag{2.6}$$

where $\phi_i = (\phi_{i1}, ..., \phi_{ir})'$. $G_i(s_t; \gamma, c)$ are often called transition functions, which are required to be at least twice differentiable and range from zero to one, s_t is the threshold variables, γ_i is the transition rate, which reflects the speed of switching from one regime to another, and c_i is the threshold value, with $c_{i-1} < c_i$ for all $i = 1, ..., m$. A comprehensive survey of recent developments of this model can be found in van Dijk et al. (2002).

The most widely used transition functions, $G(s_t; \gamma, c)$, are the logistic function given by

$$G(s_t; \gamma, c) = \frac{1}{1 + \exp(-\gamma(s_t - c))}, \tag{2.7}$$

and the exponential function given by

$$G(s_t; \gamma, c) = 1 - \exp(-\gamma(s_t - c)^2). \qquad (2.8)$$

A two-regime ($m = 2$) STAR model with a logistic (exponential) transition function is called an LSTAR (ESTAR) model (see Teräsvirta et al. (1994) and Lundbergh and Teräsvirta (2000) for applications of these models). A STAR-GARCH model allows ε_t in equation (2.6) to follow a GARCH process, as defined in (2.2)-(2.3) or, equivalently, by setting $f(x_t; \phi)$ to follow a STAR process, as defined in (2.6). Lundbergh and Teräsvirta (1999) give a comprehensive exposition of this model, but do not provide any regularity conditions for stationarity or the existence of moments, or any statistical properties. Recently, Chan and McAleer (2002) showed that the results in Ling (1999) and Ling and McAleer (2002a, b, 2003) also hold for STAR-GARCH. They showed that $E(\varepsilon_t^2) < \infty$ is sufficient for consistency and $E(\varepsilon_t^4) < \infty$ is sufficient for asymptotic normality for the QMLE of STAR-GARCH. Moreover, in the case of $p = q = 1$, the necessary and sufficient conditions for $E(\varepsilon_t^2) < \infty$ and $E(\varepsilon_t^4) < \infty$ are $\alpha + \beta < 1$, and $(\alpha+\beta)^2 + 2\alpha^2 < 1$, respectively (see Bollerslev (1986), Ling (1999), Ling and Li (1997) and Ling and McAleer (2002a, b) for further details). Furthermore, Chan and McAleer (2003) investigated the effects of outliers and extreme observations on the QMLE of the STAR-GARCH model.

A less restrictive condition, namely the log-moment condition of Nelson (1990) (see also Bougerol and Picard (1992)), is given below for the consistency and asymptotic normality of QMLE for the STAR-GARCH model with $p = q = 1$ (the proof is available upon request).

Proposition 1: *Denote $\hat{\theta}$ as the solution to the maximisation problem as defined in (2.5), with $p = q = 1$ in (2.3). Under strict stationarity and ergodicity (see Proposition 1 in Chan and McAleer (2002)), and $E(\log(\alpha_1 \eta_t^2 + \beta_1)) < 0$, it follows that $\hat{\theta}$ is consistent for θ_0 and asymptotically normal.*

Corollary 1: *If $E(\varepsilon_t^2) < \infty$, it follows that $\hat{\theta}$ is consistent for θ_0 and asymptotically normal.*

3 Empirical Results

This section examines the empirical moment conditions of STAR-GARCH models for three sets of empirical data, namely Standard and Poor's 500 Composite Index (S&P), US 3-month Treasury Bill Middle Rate returns (USTB), and the US/Australia Exchange Rate (US/AUD). Daily data for S&P are obtained from DataStream Services, with the sample period 1/1/1986 to 12/4/2000, giving 3726 observations in total. Weekly data for USTB are obtained from DataStream Services, with sample period 1/1/1986 to 30/12/1998, giving 689 observations. Daily data for US/AUD are obtained from dX EconData, with sample period 1/1/1986 to 12/4/2000, giving 3726 observations.

Of primary interest are the returns for these series, which are calculated as $r_t = \log y_t - \log y_{t-1}$.

The dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, as well as the empirical log-moment and second and fourth moment conditions for all cases, are available upon request.

3.1 S&P's Composite Index

The results show that the $\hat{\alpha}$ and $\hat{\beta}$ estimates seem to be affected greatly by the presence of outliers. When the outlier in observation 466 was removed from the rolling window, $\hat{\alpha}$ ($\hat{\beta}$) decreased (increased) from 0.106 (0.876) to 0.033 (0.959). This suggests that the outlier has a positive (negative) impact on $\hat{\alpha}$ ($\hat{\beta}$), which conforms with the empirical findings of Chan and McAleer (2003) and Verhoeven and McAleer (2002).

As the estimates are sensitive to the presence of outliers, the empirical moment conditions are subsequently affected. The movements of the empirical log-moment and second moment are similar to the movements in $\hat{\beta}$. When the outlier is removed from the rolling sample, the log-moment increased from -0.035 to -0.08, and the second moment increased from 0.982 to 0.992. This is primarily due to the fact that the outlier seemed to have a larger impact on $\hat{\beta}$ than on $\hat{\alpha}$. The mean empirical log-moment and second moment are -0.019 and 0.990, respectively.

Movements in the empirical fourth moment do not seem to be as dramatic as the log-moment and second moment. Although the results show substantial fluctuations in the fourth moment, the range of variability is narrower than for the log-moment and second moment. Despite the upward trend in the empirical fourth moment, all rolling samples satisfy the fourth moment condition, with a mean of 0.991.

Rolling estimates for S&P for ESTAR-GARCH reveal a similar pattern to LSTAR-GARCH. Movements in $\hat{\alpha}$ and $\hat{\beta}$ are almost identical to the movements in $\hat{\alpha}$ and $\hat{\beta}$ for LSTAR-GARCH: $\hat{\alpha}$ decreased from 0.107 to 0.033 when the outlier was removed from the rolling sample, while $\hat{\beta}$ increased from 0.876 to 0.959.

Not surprisingly, the movements in the empirical log-moment and second and fourth moments are also very similar to those for LSTAR-GARCH. Again, the empirical log-moment increased from -0.035 to -0.008 when the outlier was removed from the rolling sample, while the second moment increased from 0.983 to 0.992. Furthermore, movements in the empirical fourth moment are also similar to the movements in the fourth moment for LSTAR-GARCH. As in the case of LSTAR-GARCH, all rolling samples satisfy the fourth moment condition for ESTAR-GARCH. The mean log-moment and second and fourth moments are -0.019, 0.990 and 0.991, respectively.

3.2 US 3-month Treasury Bill Rate

Both $\hat{\alpha}$ and $\hat{\beta}$ moved steadily in the early rolling samples around means of 0.227 and 0.725, respectively. The inclusion of the two extreme observations (namely, 588 and 589) in the rolling samples increased $\hat{\alpha}$ from 0.231 to 0.296, while $\hat{\beta}$ decreased from 0.707 to 0.620. However, when the outlier in observation 95 was removed from the rolling sample, $\hat{\alpha}$ decreased from 0.264 to 0.148, while $\hat{\beta}$ increased from 0.639 to 0.777. This suggests that the QMLE is sensitive to extreme observations and outliers, and that the relative size of these aberrant observations would also seem to be a critical factor in determining their effects on the estimates.

All rolling samples satisfy the empirical log-moment and second moment conditions. The effects of aberrant observations on the empirical log-moment and second moment conditions are illustrated in rolling sample 87, when the log-moment decreased from -0.119 to -0.185, while the second moment decreased from 0.938 to 0.916. These changes are due primarily to the inclusion of the two extreme observations, and their effects on $\hat{\alpha}$ and $\hat{\beta}$. Similarly, the removal of the outlier in observation 95 increased both the empirical log-moment and second moment due to the effects on $\hat{\beta}$.

The first 85 rolling samples fail to satisfy the fourth moment condition for LSTAR-GARCH. However, the fourth moment begins to decline as some of the extreme observations prior to observation 95 are removed from the rolling samples, and subsequently decreased $\hat{\alpha}$ dramatically. Since the empirical fourth moment seems to be more sensitive to changes in $\hat{\alpha}$, the decline in the empirical fourth moment would to be expected.

Although $\hat{\alpha}$ and $\hat{\beta}$ exhibit greater fluctuations in the early rolling samples, these estimates vary around similar means to their LSTAR-GARCH counterparts. Moreover, the effects of the aberrant observations on the estimates of ESTAR-GARCH are identical to those of LSTAR-GARCH. The inclusion of the two extreme observations (namely 588 and 589) increased $\hat{\alpha}$ from 0.231 to 0.296, while $\hat{\beta}$ decreased from 0.707 to 0.620. Furthermore, the removal of the outlier in observation 95 decreased $\hat{\alpha}$ from 0.264 to 0.148, while $\hat{\beta}$ increased from 0.639 to 0.777. Interestingly, $\hat{\alpha}$ and $\hat{\beta}$ in ESTAR-GARCH are equal to their LSTAR-GARCH counterparts up to 3 decimal places for the rolling samples described above.

The empirical log-moment for ESTAR-GARCH reveals a similar pattern to LSTAR-GARCH. Again, the empirical log-moment decreased from -0.119 to -0.184 when the two extreme observations are included in the rolling samples, but increased from -0.178 to -0.099 when the outlier in observation 95 is removed from the rolling sample. The empirical second and fourth moments seem to be more volatile in the early rolling samples, due to the more volatile $\hat{\alpha}$ and $\hat{\beta}$ estimates of ESTAR-GARCH in the early periods. However, the effects of the aberrant observations are similar to those of LSTAR-GARCH. As in the case of LSTAR-GARCH, the first 85 rolling samples fail to satisfy the fourth moment condition due to the high $\hat{\alpha}$ estimates. However, the empirical fourth moment decreased to less than one when the outlier was removed from the rolling sample, due to its positive effects on $\hat{\alpha}$.

3.3 US/AUD Exchange Rate

The results show that $\hat{\alpha}$ declines consistently throughout the rolling samples, with two dramatic drops, namely rolling samples 329 and 392:

$\hat{\alpha}$ decreases from 0.09 to 0.064 in the first instance, and decreases further from 0.069 to 0.036 in the second.

Correspondently, $\hat{\beta}$ rises consistently throughout the rolling samples, with two dramatic increases in rolling samples 329 and 392. In fact, $\hat{\beta}$ increases from 0.860 to 0.914 in the first instance, and increases further from 0.907 to 0.959 in the second.

There is no obvious aberrant observation being removed or added in the two rolling samples. Thus, the dramatic movements do not seem to be caused by extreme observations or outliers in the data. This shows that data other than aberrant observations can cause serious changes in the estimates. This would be an interesting area for future research.

The movements in the empirical log-moment and second and fourth moments seem to mimic the movements in $\hat{\beta}$. All the rolling samples satisfy the empirical log-moment and second and fourth moment conditions.

Interestingly, $\hat{\alpha}$ and $\hat{\beta}$ for ESTAR-GARCH are often equal to their LSTAR-GARCH counterparts up to 4 decimal places. This suggests that both conditional means manage to capture the dynamics in the data. More importantly, all moment conditions are also satisfied for all the rolling samples for ESTAR-GARCH.

4 Concluding Remarks

This paper has provided a weak sufficient, or log-moment, condition for the consistency and asymptotic normality of QMLE for the STAR-GARCH(1,1) model. The condition can be extended to any non-linear time series model with GARCH(1,1) errors, subject to appropriate regularity conditions.

The effects of aberrant observations on the empirical moments were discussed through the use of rolling estimates on three data sets, namely Standard and Poor's Composite 500 Index (S&P), US 3-month Treasury Bill rate (USTB), and the exchange rate between the USA and Australia (US/AUS). The results showed that extreme observations and outliers affected the empirical moment conditions through their effects on the QMLE.

Although there have been some theoretical developments of STAR-GARCH models in recent years, the task of understanding the nature of non-linear models with conditionally heteroscedastic errors is far from complete. Lamoureux and Lastrapes (1990) examined the effects of a structural shift in the conditional variance by including dummy variables in the GARCH equation. Lundbergh and Teräsvirta (1999) extended the concept of structural change in the conditional variance by incorporating the smooth transition mechanism in the GARCH equation, known as STAR-Smooth Transition GARCH (STAR-STGARCH). Although allowing smooth transition behaviour in both the conditional mean and the conditional variance would seem to be a useful extension of STAR-GARCH, the lack of structural and statistical properties for these models has prevented their widespread use in the literature. Future research in establishing the structural and statistical properties of these models is likely to provide invaluable insights into further appropriate applications of these models.

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References

Bollerslev, T., "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 1986, *31*, 307–327.

Bougerol, P. and N.M. Picard, "Stationarity of GARCH Processes and of Some Non-negative Time Series," *Journal of Econometrics*, 1992, *52*, 115–127.

Boussama, F., "Asymptotic Normality for the Quasi-Maximum Likelihood Estimator of a GARCH Model," *Comptes Rendus de l'Academie des Sciences, Serie I*, 2000, *331*, 81–84 (in French).

Chan, F. and M. McAleer, "Maximum Likelihood Estimation of STAR and STAR-GARCH Models: Theory and Monte Carlo Evidence," *Journal of Applied*

Econometrics, 2002, *17*, 509–534.

Chan, F. and M. McAleer, "Estimating Smooth Transition Autoregressive Models with GARCH Errors in the Presence of Extreme Observations and Outliers," 2003. To appear in *Applied Financial Economics.*

Chan, K.S. and H. Tong, "On Estimating Thresholds in Autoregressive Models," *Journal of Time Series Analysis*, 1986, *7*, 179–194.

van Dijk, D., T. Teräsvirta, and P.H. Franses, "Smooth Transition Autoregressive Models - A Survey of Recent Developments.," *Econometric Reviews*, 2002, *21*, 1–47.

Elie, L. and T. Jeantheau, "Consistency in Heteroskedastic Models," *Comptes Rendus de l'Academie des Sciences, Serie I*, 1995, *320*, 1255–1258 (in French).

Engle, R.F., "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 1982, *50*, 987–1007.

Jeantheau, T., "Strong Consistency of Estimators for Multivariate ARCH Models," *Econometric Theory*, 1998, *14*, 70–86.

Lamoureux, C.G. and W.D. Lastrapes, "Persistence in Variance, Structural Change, and the GARCH Model", *Journal of Business and Economic Statistics*, 1990, *8*, 225–234.

Ling, S., "On the Probabilitistic Properties of a Double Threshold ARMA Conditional Heteroscedasticity Model," *Journal of Applied Probability*, 1999, *36*, 1–18.

Ling, S. and W.K. Li, "On Fractional Integrated Autoregressive Moving Average Time Series Models with Conditional Heteroscedasticity," *Journal of the American Statistical Association*, 1997, *92*, 1184–1194.

Ling, S. and M. McAleer, "Necessary and Sufficient Moment Conditions for the GARCH(r, s) and Asymmetric Power GARCH(r, s) Models," *Econometric Theory*, 2002a, *18*, 722–729.

Ling, S. and M. McAleer, "Stationarity and the Existence of Moments of a Family of GARCH Processes," *Journal of Econometrics*, 2002b, *106*, 109–117.

Ling, S. and M. McAleer, "Asymptotic Theory for a Vector ARMA-GARCH Model," *Econometric Theory*, 2003, *19*, 278–308.

Lundbergh, S. and T. Teräsvirta, "Modelling Economic High Frequency Time Series with STAR-STGARCH Models," *SSE/EFI Working Paper Series in Economics and Finance*, 1999, *No. 291.*

Lundbergh, S. and T. Teräsvirta, "Forecasting with Smooth Transition Autoregressive Models," *SSE/EFI Working Paper Series in Economics and Finance*, 2000, *No. 390.*

Nelson, D.B., " Stationarity and Persistence in the GARCH(1,1) Model," *Econometric Theory*, 1990, *6*, 318–334.

Terasvirta, T., "Specification, Estimation and Evaluation of Smooth Transition Autoregressive Models," *Journal of the American Statistical Association*, 1994, *89*, 208–218.

Teräsvirta, T., D. Tjøstheim, and C.W.J. Granger, *Aspects of Modelling Nonlinear Time Series*, in Engle and McFadden 1994, 2917–2957.

Tong, H., *On a Threshold Model*, in **Chen, C.H. (ed.)**, *Pattern Recognition and Signal Processing*, Amsterdam: Sijhoff and Noordoff, 1978, pp. 101–141.

Tong, H. and K.S. Lim, "Threshold Autoregressive, Limit Cycles and Data," *Journal of the Royal Statistical Society B*, 1980, *42*, 245–292.

Verhoeven, P. and M. McAleer, "Modelling Outliers and Extreme Observations for ARMA-GARCH Processes," 2002. Submitted.