A Nash Equilibrium Solution on Oligopoly Market: The Search for Nash Equilibrium Solution with Replicator Equations Derived from Gradient Dynamics on a Simplex

Aiyoshi, E and A. Maki

Faculty of Science and Technology, Keio University, Kanagawa, Japan
Faculty of Business and Commerce, Keio University, Tokyo Japan
Email: maki@fbc.keio.ac.jp)

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EXTENDED ABSTRACT

The present analysis is an application of the continuous time replicator dynamics to economics. There are various types of markets ranging from competitive markets to monopoly markets. In contemporary industrial society, oligopoly markets prevail. In this paper we discuss continuous game problems for which decision making variable for each player is bounded on a simplex by equality and non-negative constraints. Several types of problems are considered under conditions of a normalized constraint and non-negative constraints. These problems can be classified into two types by their constraints. For one type, the simplex constraint applies to the variables for each player independently such as a product allocating problem. For the other type, the simplex constraint applies to interference among all the players such as a market share problem. In this paper, we consider a game problem under the constraints of allocation of product and market share simultaneously. We assume that the problems have Nash equilibrium solution, and then we derive gradient system dynamics, which converge to the Nash equilibrium solution without violation of the simplex constraints. The story of the following models is as follows. There are three or more firms in a market. They behave so as to maximize their profits defined by the difference between their sales and cost functions with conjectural variations. In economics, there are many models concerning conjectural variation and Nash equilibrium. Lastly, the effectiveness of the derived dynamics is shown by its application to some examples. The present approach may be useful to examine the process of reaching equilibrium on oligopoly market.

1. Introduction

There are various types of markets ranging from competitive markets to monopoly markets. In contemporary industrial society, oligopoly markets prevail in manufacturing industries such as automobiles, electric appliances, PCs, etc. One of the problems facing producers is to decide on how much to produce and how to share the market among a mix of goods. The Nash equilibrium model is a useful tool for clarifying the structure of oligopoly markets. Here, we will propose a simple model of the Nash equilibrium and use a simulation method to derive an optimal solution for production decisions by rival firms.

Section 2 explains the three models in general: (i) the allocation problem of production quotas for plural products in each firm (Constraint Independent Type) (cf. Aiyoshi and Maki (2003)), (ii) the market share problem for the same product among the firm (Constraint Interference Type), (iii) double allocation problem of the product ability and the market share (Constraint Interference Type). In Section 3, we introduce the normalized Nash equilibrium solution for the profit maximization of players' functions in the models of (ii) and (iii) (cf. Rosen (1965)). Section 4 denotes how to search the Nash equilibrium solution using numerical methods. The present analysis uses the numerical method of the continuous time replicator dynamic that is used for the game-theoretic problem on ecology, group genetics and evolutionary economics (cf. Taylor and Jonker (1978). We refer Li and Basar (1987) and Bozma (1996) as algorithms for the Nash equilibrium solution although they did not describe on continuous time dynamics. Section 5 proposes a simulation model and the reports results. Finally section 6 presents some conclusions.

2. Non-cooperative Nash equilibrium model and resource allocation

In this paper, a continuous game problem with P players N strategy variables, governed by a duplicate simplex constraints, is considered. Let the p -th player's strategy variables be $\boldsymbol{x}^p = (x_1^p, \cdots, x_N^p) \in R^N$, and let $i = 1, \cdots, N,\ p = 1, \cdots, P.$ Let the variable matrix X that contains all variables as,

$$X = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^P) = \begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} x_1^1 & \cdots & x_1^P \\ \vdots & \ddots & \vdots \\ x_N^1 & \cdots & x_N^P \end{pmatrix} \tag{1}$$

(where $\boldsymbol{x}^p,\ p = 1, \cdots, P$ are column vectors and $\boldsymbol{x}_i, i = 1, \cdots, N$ are row vectors). Let the p th player's profit function be $E^p(X)$. An unconstrained game problem is formulated as

$$\max_{\boldsymbol{x}^p} E^p(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^p, \cdots \boldsymbol{x}^P),\ p = 1, \cdots, P, \tag{2}$$

where $\boldsymbol{x}^p$ is p th player's only variable, and the other players' variables $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^{p-1}, \boldsymbol{x}^{p+1} \cdots \boldsymbol{x}^P$ are unknown parameters. In the continuous game problem Eq.(2) is constrained by a linear equality and a non-negative constraint, known as a simplex constraint, as

$$\max_{\mathbf{x}^p} E^p(\mathbf{x}^1, \cdots, \mathbf{x}^p, \cdots \mathbf{x}^P) \tag{3a}$$

$$\text{subj. to} \sum_{i=1}^{N} x_i^p = a^p \tag{3b}$$

$$x_i^p \ge 0,\ i = 1, \cdots, N \tag{3c}$$

with constraints consisting of summations of column vectors of X , and

$$\max_{\mathbf{x}^p} E^p(\mathbf{x}^1, \cdots, \mathbf{x}^p, \cdots \mathbf{x}^P) \tag{4a}$$

$$\text{subj. to} \sum_{p=1}^{P} x_i^p = b_i,\ i = 1, \cdots, N \tag{4b}$$

$$x_i^p \ge 0,\ i = 1, \cdots, N, \tag{4c}$$

whose constraints consists of summations of row vectors of X . In this paper, we consider the game problem constrained by (3b) and (4b) simultaneously as

$$\max_{\mathbf{x}^p} E^p(\mathbf{x}^1, \cdots, \mathbf{x}^p, \cdots \mathbf{x}^P) \tag{5a}$$

$$\text{subj. to} \sum_{i=1}^{N} x_i^p = a^p \tag{5b}$$

$$\sum_{p=1}^{P} x_i^p = b_i,\ i = 1, \cdots, N \tag{5c}$$

$$x_i^p \ge 0,\ i = 1, \cdots, N, \tag{5d}$$

where

$$\sum_{p=1}^{P} a^p = \sum_{i=1}^{N} b_i. \tag{6}$$

As an example, P firms produce N types of products in a market. Then, p represents a firm number and i represents a product type. The first problem defined by Eq.(3) is considered a problem that each firm allocates production quotas for the N kinds of products and interferences among players exist in player's own profit function. On the other hand, the second problem formulated by

Eq.(4) is considered as a market share competition problem for the same product i among the production firms, and interferences among players is affected by player's own profit function. The last problem represented by Eq.(5) is called "a game problem with a double allocation type of constrains", namely the allocation of the production and the market share is considered simultaneously.

When we assume Nash equilibrium solutions as rational solutions for non-cooperative game problems, their properties differ, depending on their equality constraints, that is — there is a difference between the problem of Eq.(3) (Constraint Independent Type in which interference does not exist) and the problems of Eq.(4) and (5) (Constraint Interference Type in which mutual interference does exist).

3 . The Normalized Nash Equilibrium Solution of the Constraint Interference Type

Let us consider the Nash equilibrium solution $\overline{X}$ regarding the interference type problem denoted by Eq.(5) with double simplex constraints. In this case, stationary conditions for each player do not exist, unlike the situation for the constraint independent type. A maximization problem for the p th player, under the condition that other players' strategies $\overline{\boldsymbol{x}}^1,\cdots,\overline{\boldsymbol{x}}^{p-1},\overline{\boldsymbol{x}}^{p+1},\cdots,\overline{\boldsymbol{x}}^P$ are given, is expresses as

$$\max_{\mathbf{x}^p} E^p(\overline{\mathbf{x}}^1,\cdots,\overline{\mathbf{x}}^{p-1},\mathbf{x}^p,\overline{\mathbf{x}}^{p+1},\cdots\overline{\mathbf{x}}^P) \quad (7a)$$

$$\text{subj. to } \sum_{i=1}^{N} x_i^p = a^p \quad (7b)$$

$$x_i^p = b_i - \sum_{\substack{q=1\\q\neq p}}^{P} \overline{x}_i^q,\ i=1,\cdots,N \quad (7c)$$

$$x_i^p \geq 0,\ i=1,\cdots,N. \quad (7d)$$

In Eq.(7), the p th player's strategy $\boldsymbol{x}^p$ that satisfies Eq.(7c) is determined uniquely, because the other player's strategy variables are already given. There is no freedom to maximize the function of E^p, In order to define nontrivial Nash equilibrium solutions for problems of constraint interference type, we introduce the normalized Nash equilibrium solution, which has a flexibility of maximization, and whose stationary condition can be derived.

The normalized Nash equilibrium solution $\overline{X}$ regarding the constraint interference type problem is defined by relaxing the interference among players in the constraint Eq.(7c) and by considering the problem of maximizing the sum of all players' profit functions:

$$\max_X \sum_{p=1}^{P} E^p(\overline{\mathbf{x}}^1,\cdots,\overline{\mathbf{x}}^{p-1},\mathbf{x}^p,\overline{\mathbf{x}}^{p+1},\cdots\overline{\mathbf{x}}^P) \quad (8a)$$

subj. to

$$\sum_{i=1}^{N} x_i^p = a^p,\ p=1,\cdots,P \quad (8b)$$

$$\sum_{p=1}^{P} x_i^p = b_i,\ i=1,\cdots,N \quad (8c)$$

$$x_i^p \geq 0,\ i=1,\cdots,N,\ p=1,\cdots,P \quad (8d)$$

Notice that Eq.(8a) is dependent on unknown parameters $\overline{X}=(\overline{\boldsymbol{x}}^1,\cdots,\overline{\boldsymbol{x}}^P)$, and the variable $X=(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^P)$ is maximized simultaneously. Let the function $F: R^{N\times P}\times R^{N\times P}\to R^1$ be defined by

$$F(X;\overline{X})=\sum_{p=1}^{P} E^p(\overline{\boldsymbol{x}}^1,\cdots,\overline{\boldsymbol{x}}^{p-1},\boldsymbol{x}^p,\overline{\boldsymbol{x}}^{p+1},\cdots\overline{\boldsymbol{x}}^P). \quad (9)$$

We define $\overline{X}$ as the local normalized Nash equilibrium solution of the constraint interference type problem Eq.(5), when there exists a neighbourhood $B(\overline{X})\subseteq R^{N\times P}$ of $\overline{X}$ such that the following inequality holds:

$$F(\overline{X};\overline{X})\geq F(X;\overline{X})\ \forall X\in B(\overline{X})\cap S, \quad (10)$$

where

$$S=\{X \mid X \text{ satisfies Eq.(8b), (8c), (8.d)}\}.$$

Note that the normalized Nash equilibrium solution is not a solution of a simple maximization problem of the sum of all players' profit function F, but the maximum point of F with respect to the variable X in F, given the value $\overline{X}$ in F as a parameter, namely, it is defined as a fixed point of the maximization operation.

4. Dynamics to Search the Nash Equilibrium Solution of Constraint Interference Type

Firstly, in the constraint independent type problem Eq.(3), let us consider a gradient system in which the p th player's strategy $\boldsymbol{x}^p$ moves to a direction to locally maximize the profit function $E^p(X)$:

$$\frac{d\mathbf{x}^p(t)}{dt}=\nabla_{\mathbf{x}^p}E^p(X(t)) \quad (11a)$$

$$\nabla_{\mathbf{x}^p}E^p(X)=\left(\frac{\partial E^p(X)}{\partial x_1^p},\cdots,\frac{\partial E^p(X)}{\partial x_N^p}\right). \quad (11b)$$

A trajectory of this dynamics violates the equality constraint Eq.(3b). Therefore, we can derive a model that orthogonally projects $\nabla_{x^p} E^p(X)$ to the hyperplane defined by Eq.(3b) by application of the gradient projection method with a coefficient matrix for Eq.(3b) $A = (1, \cdots, 1)$. However, even if the search can be executed, the projected vector violates the non-negative constraint Eq.(3c) if the search point lies on a boundary of the region defined by the non-negative constraint Eq.(3c). Hence we derive a new projection operator taking the non-negative constraint into consideration. So, we introduce a variable metric which increase as the search point approaches to the boundary of non-negative constraint and becomes infinity on the boundary according to the $N \times N$ matrix:

$$M(\boldsymbol{x}^p) = \mathrm{diag}(1/x_i^p), \ p = 1, \cdots, P. \tag{12}$$

Each component of the direction vector $\nabla_{x^p} E^p(X(t))$ is reduced by multiplication by $M(\boldsymbol{x}^p)$, as the search point approaches the boundary. The operator is defined as an $N \times N$ variable metric projection matrix proposed by Faybusovich (1991) in terms of the inverse matrix of Eq. (12) and the coefficient matrix $A = (1, \cdots, 1)$:

$$Q_A^M(\mathbf{x}^p) = I - M^{-1}(\mathbf{x}^p) A^T (A M^{-1}(\mathbf{x}^p) A^T)^{-1} A$$
$$= I - \mathrm{diag}(x_i^p) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \left(\sum_{i=1}^N x_i^p \right)^{-1} (1, \cdots, 1)$$
$$= I - (x^p, \cdots, x^p). \tag{13}$$

Multiplying the gradient vectors regarding all variables of the p th player in Eq.(11b) by the inverse variable metric matrix Eq.(12) and multiplying by the variable metric projection matrix Eq.(13), we obtain a dynamics which does not violate the simplex constraint and which takes the non-negative constraint into consideration:

$$\frac{d\boldsymbol{x}^p(t)}{dt} = Q_A^M(\boldsymbol{x}^p(t)) M^{-1}(\boldsymbol{x}^p(t)) \nabla_{x^p} E^p(X(t)) \tag{14}$$

Expressing in terms of components, we obtain

$$\frac{dx_i^p(t)}{dt} = x_i^p(t) \left\{ \frac{\partial E^p(X(t))}{\partial x_i^p} - \sum_{j=1}^N \frac{\partial E^p(X(t))}{\partial x_j^p} x_j^p(t) \right\},$$
$$i = 1, \cdots, N. \tag{15}$$

Here, letting x_i^p be the "fraction of individuals in the p th species choosing the i th strategy", and $f_i^p(X) = \partial E^p(X) / \partial x_i^p$ be the "expected profit value which an individual in the p th species using the i th strategy obtains per unit time". Then, Eq.(15) is a replicator dynamics to solve an evolutionary game problem of P species employing mixed strategies.

Next, we derive a dynamics to search for the normalized Nash equilibrium solution of constraint interference type Eq.(4) using a similar procedure as the constraint independent type mentioned above. Let us consider a gradient vector

$$\left(\frac{\partial E^1(X)}{\partial x_i^1}, \cdots, \frac{\partial E^P(X)}{\partial x_i^P} \right) \tag{16}$$

with respect to the i th component vector $\boldsymbol{x}_i$ in Eq.(9). Note that the partial derivative of x_i^p is only employed with respect to E^p, because $\partial E^q(X) / \partial x_i^p = 0, q \neq p$ in Eq.(9). Multiplying Eq.(16) by an inverse $P \times P$ variable metric matrix

$$M^{-1}(\boldsymbol{x}_i) = \mathrm{diag}(x_i^p), i = 1, \cdots, N, \tag{17}$$

we obtain a $P \times P$ variable metric projection matrix operator which projects each component of Eq.(16) to its simplex constraint (Eq.(4b)), by taking the inverse matrix of Eq.(17) as the matrix for the metric:

$$Q_A^M(\mathbf{x}_i) = I - A(A^T M^{-1}(\mathbf{x}_i)A)^{-1} A^T M^{-1}(\mathbf{x}_i)$$

$$= I - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \left(\sum_{p=1}^{P} x_i^p\right)^{-1} (1,\cdots,1)\,\mathrm{diag}(x_i^p)$$

$$= I - \begin{pmatrix} \mathbf{x}_i \\ \vdots \\ \mathbf{x}_i \end{pmatrix}. \qquad (18)$$

Using Eq.(17) and Eq.(18), we can consider a gradient dynamics on the region defined by the simplex constraint:

$$\frac{d\mathbf{x}_i(t)}{dt} = \left(\frac{\partial E^1(X(t))}{\partial x_i^1}, \cdots, \frac{\partial E^P(X(t))}{\partial x_i^P}\right) \cdot M^{-1}(\mathbf{x}_i(t))\, Q_A^M(\mathbf{x}_i(t)) \qquad (19)$$

Expressing in terms of components, we obtain

$$\frac{dx_i^p(t)}{dt} = x_i^p(t)\left\{\frac{\partial E^p(X(t))}{\partial x_i^p} - \sum_{q=1}^{P} \frac{\partial E^q(X(t))}{\partial x_j^q} x_j^q(t)\right\}, \quad i = 1,\cdots,N \quad . \qquad (20)$$

Here, letting $f_i^p(X) = \partial E^p(X)/\partial x_i^p$, $N = 1$, and x^p be the "fraction of population (of a species) which choose the p th strategy", and $f^p(\boldsymbol{x})$ be the "expected profit value which obtained per unit time by an individual employing the p th strategy". Then, Eq.(20) is a replicator dynamics to solve an evolutionary game problem with single species.

Lastly, we investigate a dynamics to search for the normalized Nash equilibrium solution of constraint interference type Eq.(5), in which double constraints with respect to allocations of the production ability and the market share are imposed simultaneously. In order to apply the results of the above constraint interference type to the double constraint case directly, we transform the $N \times P$ matrix variable X to the $N \times P$ dimensional column vector variable as $\boldsymbol{X} = (\boldsymbol{x}^{1T},\cdots,\boldsymbol{x}^{PT})^T$, and reform the double constraints Eq.(5b), (5c) as a linear equality constraint of vector-matrix form $\boldsymbol{AX} = \boldsymbol{c}$ with a $(P+N)\times(N\times P)$ coefficient matrix $\boldsymbol{A}$. That is,

$$\mathbf{X} = (\mathbf{x}^{1^T},\cdots,\mathbf{x}^{P^T})^T, \qquad (21a)$$

$$\mathbf{A} = \begin{bmatrix} 1\cdots1 & & & \\ & 1\cdots1 & & \\ & & \ddots & \\ & & & 1\cdots1 \\ 1 & 1 & \cdots & 1 \\ \ddots & \ddots & & \ddots \\ 1 & 1 & & 1 \end{bmatrix}, \qquad (21b)$$

$$\mathbf{c} = (a^1,\cdots,a^P,b_1,\cdots,b_N)^T. \qquad (21c)$$

Here, we have to notice that an arbitrary element of equality $\boldsymbol{AX} = \boldsymbol{c}$ is satisfied, because the sum of Eq.(8b) is equal to the sum of Eq.(8c) under the balancing condition of supplies and demands in Eq.(6), and then $\mathrm{rank}\,\boldsymbol{A} = N + P - 1$. Let $\overline{\boldsymbol{A}}$ be the $(P+N-1)\times(N\times P)$ matrix in which an arbitrary row of matrix $\boldsymbol{A}$ is deleted. Trough a similar way to constraint interference type Eq.(4), we can propose a dynamics to search for the normalized Nash equilibrium solution of a game problem with a double allocation type of interference constraints as follows:

$$\frac{d\boldsymbol{X}(t)}{dt} = Q_{\overline{A}}^M(\boldsymbol{X}(t))M^{-1}(\boldsymbol{X}(t))\nabla F(\boldsymbol{X}(t);\boldsymbol{X}(t)), \qquad (22)$$

where

$$\frac{d\mathbf{X}(t)}{dt} = \begin{pmatrix} d\mathbf{x}^1(t)/dt \\ \vdots \\ d\mathbf{x}^P(t)/dt \end{pmatrix}, \qquad (23a)$$

$$Q_{\overline{\mathbf{A}}}^M(\mathbf{X}) = I - M^{-1}(\mathbf{X})\overline{\mathbf{A}}^T(\overline{\mathbf{A}}M^{-1}(\mathbf{X})\overline{\mathbf{A}}^T)^{-1}\overline{\mathbf{A}}, \qquad (23b)$$

$$M^{-1}(\mathbf{X}) = \operatorname{diag}(1/x_i^p) \quad (N\times P)\times(N\times P)\text{matrix} \tag{23c}$$

$$\nabla F(\mathbf{X};\mathbf{X}) = \begin{pmatrix} \nabla_{\mathbf{x}^1} E^1(\mathbf{x}^1,\cdots,\mathbf{x}^P) \\ \vdots \\ \nabla_{\mathbf{x}^P} E^P(\mathbf{x}^1,\cdots,\mathbf{x}^P) \end{pmatrix}. \tag{23d}$$

Here, we regret that $(N\times P)\times(N\times P)$ variable metric projection matrix $Q_{\bar{A}}^M(\boldsymbol{X})$ cannot be expressed by a simple formula as Eq.(13) or Eq.(18), because the inverse $(\bar{\boldsymbol{A}}M^{-1}(\boldsymbol{X})\bar{\boldsymbol{A}}^T)^{-1}$ cannot be formulated explicitly.

5. Simulations of Dynamic to the Normalized Nash Equilibrium for the Double Resource Allocation Problem

For simplicity, we consider a three-person (P=3) game with three products (N=3). Even in the simplest model, there is no loss of generality in the model described in section 2. As a concrete example, there are three automobile companies, and three of them produce three types of automobiles, namely ordinary, medium and luxury cars. The decision variables are $\boldsymbol{x}^1=(x_1^1,x_2^1,x_3^1)^T$, $\boldsymbol{x}^2=(x_1^2,x_2^2,x_3^2)^T$ and $x^3=(x_1^3,x_2^3,x_3^3)^T$ where suffix indicates product and superfix indicates firm. The profit functions of each firm are

$$E^p(X) = -(p+1)\sum_{i=1}^{N}(x_i^p - 1/(i+1))^2 - \sum_{i=1}^{N}\sum_{\substack{q=1\\q\neq p}}^{N}\theta_{pqi}x_i^p x_i^q, \tag{24}$$

where θ_{pqi} is a loss parameter suffered by the i-product when p-player produces x_i^p and q-player produces x_i^q. In economics of firms, gain is the corporate profit and loss is various kinds of conjectural costs. The constraints are the double allocation type of the production ability and the market share as Eq.(5b) and Eq (5c) and set $a^1=a^2=a^3=1$, , $b_1=b_2=b_3=1$ for simplicity. Concretely, for firm 1, the gain from products 1, 2 and 3 are indicated respectively as:

$$f_1^1(x_1^1) = -2(x_1^1-0.5)^2,\quad f_2^1(x_2^1) = -2(x_2^1-0.333)^2,\quad f_3^1(x_3^1) = -2(x_3^1-0.25)^2. \tag{25}$$

The difference in the functions f_i^p is due to that of production technology for products and firms. The profit function $E^1(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3)$ for firm 1 is specified as:

$$E^1(\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3) = f_1^1(x_1^1)+f_2^1(x_2^1)+f_3^1(x_3^1) - (\theta_{121}x_1^1x_1^2+\theta_{122}x_2^1x_2^2+\theta_{123}x_3^1x_3^2 + \theta_{131}x_1^1x_1^3+\theta_{132}x_2^1x_2^3+\theta_{133}x_3^1x_3^3)+2,$$

where $\theta_{111},\theta_{112}$ and θ_{113} are assumed to be zero. For firm 2, the gain functions for products 1, 2 and 3 are, respectively:

$$f_1^2(x_1^2) = -3(x_1^2-0.5)^2,\quad f_2^2(x_2^2) = -3(x_2^2-0.333)^2,\quad f_3^2(x_3^2) = -3(x_3^2-0.25)^2. \tag{26}$$

The profit function $E^2(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3)$ for firm 2 is specified as:

$$E^2(\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3) = f_1^2(x_1^2)+f_2^2(x_2^2)+f_3^2(x_3^2) - (\theta_{211}x_1^2x_1^1+\theta_{212}x_2^2x_2^1+\theta_{213}x_3^2x_3^1 + \theta_{231}x_1^2x_1^3+\theta_{232}x_2^2x_2^3+\theta_{233}x_3^2x_3^3)+2,$$

where $\theta_{221},\theta_{222}$ and θ_{223} are assumed to be zero.

The profit function, $E^3(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3)$, for firm 3 is specified as: $f_1^2(x_1^2) = -4(x_1^3-0.5)^2$,

$$f_2^2(x_2^2) = -4(x_2^3-0.333)^2,\quad f_3^2(x_3^2) = -4(x_3^3-0.25)^2. \tag{27}$$

$$E^3(\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3) = f_1^3(x_1^3)+f_2^3(x_2^3)+f_3^3(x_3^3) - (\theta_{311}x_1^3x_1^1+\theta_{312}x_2^3x_2^1+\theta_{313}x_3^3x_3^1 + \theta_{321}x_1^3x_1^2+\theta_{322}x_2^3x_2^2+\theta_{323}x_3^3x_3^2)+2,$$

where $\theta_{331},\theta_{332}$ and θ_{333} are assumed to be zero.

In the first simulation, we fixed the eighteen values of θ_{pqi}'s as the same as 0.5 and get the normalized Nash equilibrium solution for the decision variables of $X=(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3)$. Table 1 indicates the change in the normalized Nash equilibrium value for $X=(\boldsymbol{x}^1,\boldsymbol{x}^2,\boldsymbol{x}^3)$ from the initial values to the converged values.

Table 1. Converged values for X from $X(0)$

Initial values of $X(0)$	→ Converged values of X
0.001 0.001 0.998	0.037 0.418 0.545
0.001 0.998 0.001	0.428 0.306 0.266
0.998 0.001 0.001	0.535 0.276 0.189

In the second simulation, we changed the conjectural variation of firm p against q from 4.0 to 3.0 due to product differentiation. Table 2 indicates the Nash equilibrium solution for $X = (\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ by changing the parameters of θ_{pqi}'s. From Table 2, we understand the changes of the product mix for firms 1, 2 and 3 due to changes in the parameter of θ_{pqi}'s. In the oligopoly market, the conjectural variation plays an important role in determining the share of the products within a firm and among firms.

Table 2. Converged values for X from $X(0)$

(a) $\theta_{1qi} = 4.0,\ \theta_{2qi} = 4.0\ \theta_{3qi} = 4.0$

Initial values of $X(0)$	→ Converged values of X
0.001 0.001 0.998	0.75 0.25 0.00
0.001 0.998 0.001	0.25 0.75 0.00
0.998 0.001 0.001	0.00 0.00 1.00

(b) $\theta_{1qi} = 3.0,\ \theta_{2qi} = 4.0\ \theta_{3qi} = 4.0$

Initial values of $X(0)$	→ Converged values of X
0.001 0.001 0.998	1.00 0.00 0.00
0.001 0.998 0.001	0.00 1.00 0.00
0.998 0.001 0.001	0.00 0.00 1.00

(c) $\theta_{1qi} = 4.0,\ \theta_{2qi} = 3.0\ \theta_{3qi} = 4.0$

Initial values of $X(0)$	→ Converged values of X
0.001 0.001 0.998	1.00 0.00 0.00
0.001 0.998 0.001	0.00 1.00 0.00
0.998 0.001 0.001	0.00 0.00 1.00

(d) $\theta_{1qi} = 4.0,\ \theta_{2qi} = 4.0\ \theta_{3qi} = 3.0$

Initial values of $X(0)$	→ Converged values of X
0.001 0.001 0.998	0.75 0.25 0.00
0.001 0.998 0.001	0.24 0.76 0.00
0.998 0.001 0.001	0.00 0.00 1.00

6. Conclusion

Using the Nash equilibrium simulation model, we can generate various kinds of optimal paths for changing the conjecture between two firms. In the simulation the share of products produced varies according to changes in conjecture. To test the validity of the Nash equilibrium model, we need to construct an empirical model using existing data for oligopoly markets by estimating profit functions. The conjectural factor is calculated by the gap between observed data and estimated values.

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