Portfolio Single Index (PSI) Multivariate Volatility Models

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EXTENDED ABSTRACT

The paper introduces the structure of parsimonious Portfolio Single Index (PSI) multivariate conditional and stochastic constant correlation volatility models, and methods for estimation of the underlying parameters. These multivariate estimates of volatility can be used for more accurate portfolio and risk management, to enable efficient forecasting of Value-at-Risk (VaR) thresholds, and to determine optimal Basel Accord capital charges. A parsimonious portfolio single index approach for modelling the conditional and stochastic covariance matrices of a portfolio of assets is developed, and estimation methods for the conditional and stochastic volatility models are discussed.

1 Introduction

The paper introduces the structure of parsimonious Portfolio Single Index (PSI) multivariate conditional and stochastic constant correlation volatility models, and methods for estimation of the underlying parameters. These multivariate estimates of volatility can be used for purposes of more accurate portfolio and risk management, to enable efficient forecasting of Value-at-Risk (VaR) thresholds, and to determine optimal Basel Accord capital charges (a comprehensive discussion of alternative univariate and multivariate, conditional and stochastic, financial volatility models for calculating VaR is given in McAleer (2005)).

The plan of the paper is as follows. Section 2 presents the portfolio single index approach to model the conditional and stochastic covariance matrices of a portfolio of assets parsimoniously. Estimation methods for the conditional and stochastic volatility models are discussed in Section 3.

2 Portfolio Single Index Approach

2.1 Portfolio Model

Let the returns on $m\,(\geq 2)$ financial assets be given by

$$y_{it} = \mu_{it} + \varepsilon_{it}, \quad i = 1, \ldots, m, \quad t = 1, \ldots, T,$$

or

$$y_t = \mu_t + \varepsilon_t, \tag{1}$$

where y_t, μ_t and ε_t are m dimensional column vectors,

$$\mu_t = E\left(y_t \mid \Im_{t-1}\right),$$

and $\Im_t$ is the past information available at time t. The return of the portfolio consisting of m assets is denoted as

$$y_{P,t} = w'y_t = w'\mu_t + w'\varepsilon_t, \tag{2}$$

where

$$w = \left(w_1, \ldots, w_m\right)'$$

denotes the portfolio weights, such that

$$\sum_{i=1}^{m} w_i = 1.$$

For the returns to the portfolio, the conditional mean vector and disturbance of the portfolio are defined by

$$\mu_{P,t} = E\left(y_{P,t} \mid \Im_{t-1}\right) = w'\mu_t$$

and

$$\varepsilon_{P,t} = y_{P,t} - \mu_{P,t},$$

respectively. In order to consider the volatility of the portfolio, it is necessary to model the conditional and stochastic covariance matrices Q_t and Σ_t, respectively.

2.2 Conditional Volatility

Consider the conditional covariance matrix of y_t, which is given as:

$$Q_t = V\left(y_t \mid \Im_{t-1}\right) = E\left(\varepsilon_t \varepsilon_t' \mid \Im_{t-1}\right), \tag{3}$$

and the conditional volatility of the portfolio, which is given by

$$h_{P,t} = V\left(y_{P,t} \mid \Im_{t-1}\right) = w'Q_t w.$$

In the framework of multivariate GARCH models, the constant conditional correlation (CCC) model of Bollerslev (1990) abandons significant information as each component of y_t

follows a univariate GARCH(1,1) process, and hence does not capture the effects of the remaining $m-1$ assets. On the other hand, more general specifications, such as the VARMA-GARCH model of Ling and McAleer (2003) and the BEKK (Baba, Engle, Kraft and Kroner) model of Engle and Kroner (1995) suffer from the fact that the number of parameters increases significantly as the number of variables increases. This can cause serious problems for convergence of the appropriate estimators, especially for a portfolio with a large number of assets.

As an illustration, consider the VARMA-AGARCH model of Hoti, Chan and McAleer (2002). This model is an asymmetric extension of the VARMA-GARCH model of Ling and McAleer (2003), and is given by

$$\varepsilon_t = D_t \eta_t, \quad \eta_t : iid(0, \Gamma), \tag{4}$$

$$Q_t = D_t \Gamma D_t \tag{5}$$

$$H_t = \omega + \sum_{k=1}^{q} \left[A_k + C_k \operatorname{diag}\{d_{t-k}^{-}\} \right] \times (\varepsilon_{t-k} \circ \varepsilon_{t-k}) + \sum_{l=1}^{p} B_l H_{t-l} \tag{6}$$

where

$$D_t = \operatorname{diag}\{h_t\},$$

$$H_t = (h_{1t}, \ldots, h_{mt})',$$

$\operatorname{diag}\{x\}$ for any vector x denotes a diagonal matrix with x along the diagonal, and '$\circ$' denotes the Hadamard product of two identically-sized matrices or vectors, which is computed simply by element-by-element multiplication.

The vector given by

$$d_t^{-} = (d_{1t}^{-}, \ldots, d_{mt}^{-})'$$

denotes a set of indicator variables, where d_{it}^{-} takes the value one if ε_{it} is negative, and zero otherwise. For estimation of the parameters, Γ is the positive definite correlation matrix of η_t, that is,

$$E(\eta_t \eta_t') = \Gamma,$$

$$\omega = (\omega_1, \ldots \omega_m)',$$

and A_k, B_l and C_k are $m \times m$ matrices, with typical elements $\alpha_{ij,k}$, $\beta_{ij,l}$ and $\gamma_{ij,k}$, respectively.

The model of Ling and McAleer (2003) assumes $C_k = 0$ for all k in equation (6). Another special case of the VARMA-AGARCH model is the CCC model of Bollerslev (1990), which is obtained by setting all the off-diagonal elements of A_k and B_l, and all the elements of C_k, to zero. Thus, when $p = q = 1$, $m = 4$ implies that the number of parameters to be estimated in equation (6) is 52. The regularity conditions and asymptotic properties of the estimators for the various models given above are developed in Ling and McAleer (2003) and Hoti et al. (2002). These regularity conditions are extensions of the univariate results given in Ling and McAleer (2002a, b).

There are other approaches for modelling Q_t, such as the dynamic conditional correlation (DCC) model suggested by Engle (2002). However, this does not affect the ways in which a portfolio can be transformed to a single index using the methods described above.

As an intermediate approach, namely one that incorporates volatility spillover effects parsimoniously, this paper proposes the portfolio single index model, which is given as follows:

$$H_t = \omega + \sum_{k=1}^{q} (\alpha_k + \gamma_k \circ d_{t-k}^{-}) \circ (\varepsilon_{t-k} \circ \varepsilon_{t-k})$$

$$+\sum_{l=1}^{p}\beta_l \circ H_{t-l}+\sum_{s=1}^{r}\left(\delta_s\varepsilon_{P,t-s}^2+\lambda_s h_{P,t-s}\right), \quad (7)$$

where

$$\alpha_k=\left(\alpha_{1,k},\ldots,\alpha_{m,k}\right)',$$
$$\beta_l=\left(\beta_{1,l},\ldots,\beta_{m,l}\right)',$$

$$\gamma_k=\left(\gamma_{1,k},\ldots,\gamma_{m,k}\right)',$$

$$\delta_s=\left(\delta_{1,s},\ldots,\delta_{m,s}\right)'$$

and

$$\lambda_s=\left(\lambda_{1,s},\ldots,\lambda_{m,s}\right)'.$$

It should be noted that, for the portfolio returns, $\varepsilon_{P,t}=w'\varepsilon_t$ and $h_{P,t}=w'Q_t w$. The model in equations (1)-(5) and (7) will be called the Portfolio Single Index GARCH (PSI-GARCH or, equivalently, Ψ-GARCH) model. In the Ψ-GARCH model, the conditional volatility for each y_{it} may be considered as a combination of the GJR model and the portfolio spillover effects. When γ_k, δ_s and λ_s are all equal to zero, the model reduces to CCC. The asymmetry arises when γ_k is not a null vector, while non-zero δ_s and λ_s capture the portfolio spillover effects. When $p=q=r=1$ and $m=4$, the number of parameters to be estimated in equation (6) is 24. Compared with equation (6), this is a significant reduction in the number of parameters, while retaining spillover effects.

Based on the concept of weak and strong GARCH processes, as defined in Drost and Nijman (1993), Nijman and Sentana (1996) show that a linear combination of variables generated by a multivariate GARCH process is also a weak GARCH process. Thus, the Ψ-GARCH model developed in this paper is a weak GARCH process.

2.3 Stochastic Volatility

Now we turn to the stochastic covariance matrix given by Σ_t. For purposes of convenience and parsimony, we assume the presence of constant correlations in the model, such that:

$$\Sigma_t=D_t\Gamma D_t, \quad (8)$$

where

$$D_t=\operatorname{diag}\left\{0.5\exp\left(\alpha_t\right)\right\},$$

$$\alpha_t=\left(\alpha_{1t},\ldots,\alpha_{mt}\right)',$$

and 'exp' denotes the operator for vectors which performs element-by-element exponentiation. In the model, $\exp\left(\alpha_{it}\right)$ denotes the stochastic volatility for y_{it}, while the volatility for the portfolio is defined as

$$\sigma_{P,t}^2=\exp\left(\alpha_{P,t}\right), \quad (9)$$

where

$$\begin{aligned}\alpha_{P,t}&=\log\left|w'D_t\Gamma D_t w\right|\\&=\log\left|ww'\right|+\log\left|\Gamma\right|+\log\left|D_t^2\right| \quad (10)\\&=\log\left|w'\Gamma w\right|+\sum_{i=1}^{m}\alpha_{it}.\end{aligned}$$

Hence, the log-volatility of the portfolio is defined as a constant term plus the sum of the log-volatility of each asset in the portfolio.

Before developing the new stochastic volatility model, consider the VAR(p)-ASV model, as follows:

$$\varepsilon_t=D_t\eta_t, \quad \eta_t \sim N\left(0,\Gamma\right), \quad (11)$$

$$\alpha_{t+1} = \omega + \sum_{l=1}^{p} \Phi_l \alpha_{t+1-l} + \xi_t, \tag{12}$$

$$\xi_t \sim N\left(0, \Sigma_\xi\right),$$

$$E\left(\xi_t' \eta_t\right) = diag\left\{\rho_1 \sigma_{\xi,11}^{1/2}, \ldots, \rho_m \sigma_{\xi,mm}^{1/2}\right\}, \tag{13}$$

where $\Sigma_\xi = \left\{\sigma_{\xi,ij}\right\}$. For convenience, normality is assumed for the VAR(p)-ASV model. A multivariate *t* distribution is also assumed for η_t, as in Harvey, Ruiz and Shephard (1994). Non-zero values of $\rho_1, \ldots, \rho_m$ refers to the existence of leverage in the volatility of each asset. In the VAR(p)-ASV model, each log-volatility is affected by the past log-volatilities of the other $m-1$ assets through Φ_l, and also has a contemporaneous effect between the log-volatilities via Σ_ξ.

Assuming $p = 1$, $\rho_1 = \cdots = \rho_m = 0$, and Φ_1 is the diagonal matrix, we have the model proposed by Harvey et al. (1994). Based on the MSV model of Harvey et al. (1994), Asai and McAleer (2005b) considered non-zero values of $\rho_1, \ldots, \rho_m$, as in equation (13), and proposed the MCL estimation procedure for an asymmetric multivariate stochastic volatility model with a constant correlation structure.

As a closed form expression for the likelihood function of SV models does not exist, estimation of the parameters in SV models is undertaken using numerical methods by evaluating the likelihood (through numerical integration) or by simulation methods. The Monte Carlo Likelihood (MCL) methods proposed by Durbin and Koopman (1997) and Sandmann and Koopman (1998) are based on importance sampling. Although the econometrics and statistics literature has tended to focus on the Bayesian Markov Chain Monte Carlo (MCMC) method, the MCL method has the advantage in being computationally fast (in comparison with most other simulation methods) and relatively easy to implement.

A similar discussion about conditional volatility can be applied to equation (12). If we set all the off-diagonal elements of Φ_l and Σ_ξ to zero, then each y_{it} collapses to the simple ASV model of Harvey and Shephard (1996). On the other hand, the above VAR(p)-ASV model has many parameters to be estimated. When $p = 1$, $m = 4$ implies that the number of parameters in equation (12) is 30.

For the Portfolio Single Index MSV (PSI-MSV or, equivalently, Ψ-MSV) model, the log-volatility is defined as follows:

$$\alpha_{t+1} = \omega + \sum_{l=1}^{p} \phi_l \circ \alpha_{t+1-l}$$

$$+ \sum_{s=1}^{r} f\left(\varepsilon_{P,t+1-s}, \alpha_{P,t+1-s}\right) + \xi_t, \tag{14}$$

$$\xi_t \sim N\left(0, diag\left\{\sigma_{\xi,11}, \ldots, \sigma_{\xi,mm}\right\}\right),$$

where

$$\phi_l = \left(\phi_{1,l}, \ldots, \phi_{m,l}\right)'$$

and

$f\left(y_{P,t}, \alpha_{P,t}\right)$ is a function of the information contained in the portfolio. Neglecting $\varepsilon_{P,t}$, if we assume that

$$f_1\left(\alpha_{P,t+1-s}\right) = \lambda_s \alpha_{P,t-s}, \tag{15}$$

and $p = r$, where

$$\lambda_s = \left(\lambda_{1,s}, \ldots, \lambda_{m,s}\right)',$$

then we can obtain the off-diagonal elements of Φ_l under appropriate restrictions, since the log-volatility of the portfolio is defined as equation (10). If we assume that

$$f_2\left(\varepsilon_{P,t+1-s}\right)=\delta_{1,s}\varepsilon_{P,t-s}+\delta_{2,s}\left|\varepsilon_{P,t-s}\right|, \qquad (16)$$

where

$$\delta_{1,s}=\left(\delta_{1,1,s},\ldots,\delta_{1,m,s}\right)',$$

$$\delta_{2,s}=\left(\delta_{2,1,s},\ldots,\delta_{2,m,s}\right)',$$

then we can capture the asymmetric effects from shocks in the portfolio. Asai and McAleer (2005a) developed and discussed this type of asymmetry in detail. Other specifications, including a combination of (15) and (16), can be considered. However, returning to the purpose of the PSI approach, we concentrate on equation (16). In this case, the number of parameters in equations (14) and (16) reduces dramatically to 20 when $p=r=1$ and $m=4$.

Next, consider the model that $\varepsilon_{P,t}$ in (16) is replaced by the returns of the market portfolio, say $y_{M,t}$. For this model, each element of volatility is determined by using the information of the market portfolio instead of the portfolio discussed in the paper. Although this idea has intuitive appeal, we will consider them separately.

It should be stressed that the Ψ-MSV model in (11), (13), (14) and (16) has been developed as an intermediate approach for incorporating the information from the other assets in the portfolio. It is a separate matter altogether whether to use the information from market returns to supplement the information that is contained in the portfolio.

3 Estimation

3.1 Conditional Volatility Model

Under the assumption of normality of the conditional distribution of the standardized residuals, we can obtain the parameters by maximum likelihood (ML) estimation, as follows:

$$\hat{\theta}=\arg\max\sum_{t=1}^{T}l_t,$$

where

$$l_t=-\frac{1}{2}\log h_{P,t}-\frac{\varepsilon_{P,t}^2}{2h_{P,t}}, \qquad (17)$$

and θ denotes the vector of parameters to be estimated in the conditional log-likelihood function. If the assumption of normality does not hold for the standardized residuals, equation (17) is defined as the Quasi-maximum likelihood estimator (QMLE).

3.2 Stochastic Volatility Model

We focus on the Ψ-MSV model in (11), (13), (14) and (16). Estimation of the parameters in MSV models is computationally demanding, even for $m=2$. As the PSI approach presented in the paper concentrates information contained in the other assets into a single index, it enables the use of a computationally efficient method, as described below. Importantly, given the structure of the model, we can estimate the parameters for each asset, namely ω_i, $\phi_{i,l}$, $\sigma_{\xi,ii}$, ρ_i, $\delta_{1,i,s}$, $\delta_{2,i,s}$ and the parameters for μ_{it}, neglecting the remaining assets. There are numerous ways in which MSV models can be estimated, such as Monte Carlo Likelihood (MCL) method, or the Bayesian Markov Chain Monte Carlo (MCMC) method proposed by Jacquier, Polson and Rossi (1994). On the basis of Monte Carlo experiments, Sandmann and Koopman (1998) showed that the finite sample properties of the two estimators were very similar. McAleer (2005) discusses these and other methods for estimating univariate and multivariate SV models.

The recommended two step estimation method is as follows:

(1) For each financial asset, y_{it}, obtain a consistent estimate of ε_{it}, $\hat{\varepsilon}_{it}$, to calculate the portfolio shocks, $\hat{\varepsilon}_{P,t}$;
(2) For each financial asset, estimate the parameters for each volatility by using $\hat{\varepsilon}_{P,t}$ to calculate an estimate of α_{it}, $\hat{\alpha}_{it}$;

(3) he standardized residuals, $\hat{\eta}_t = \hat{\varepsilon}_t \exp(-0.5\hat{\alpha}_t)$, can be used to obtain an estimate of the correlation matrix, Γ.

After obtaining the two step estimates using the approach given above, we can obtain the estimate of $\alpha_{P,t}$ based on equation (10).

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