Stochastic Volatility with Long–Range Dependence

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EXTENDED ABSTRACT

Since Merton (1969), the description of a contingent claim as a Brownian motion is commonly accepted. Thus an option price, a future price, a share price, a bond price, interest rates etc., can be modelled with a Brownian motion. In summary, any financial series which present value depends on only a few previous values, may be modelled with a continuous–time diffusion–type process. The general diffusion equation is given by,

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t), \qquad (1)$$

where $\mu(X(t))$ is the drift function and $\sigma(X(t))$ is the volatility function of the process. There is a vast list of references related to developments on the short–term interest rate as a stochastic diffusion. For instance: a) Vacisek (1977) studied equation (1) for a mean–reverting drift function and a constant volatility; b) Cox, Ingersoll and Ross (1985) posed the CIR model which contains the square root of $X(t)$ as part of the diffusion function.

There exist financial data with long–range dependence (LRD). The typical data with this property is obtained as aggregation of several processes of type (1). For instance, portfolios or indexes such as the S&P 500, Nikkei, FTSE, etc. It is important to exploit this property as information imbedded in the data can be used to find arbitrage opportunities. Processes with LRD are modelled by,

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB_\beta(t). \qquad (2)$$

Equation (2) differs from equation (1) in the diffusion term: the classical Brownian motion is substituted by a fractional Brownian motion of the form $B_\beta(t) = \int_0^t \frac{(t-s)^\beta}{\Gamma(1+\beta)}dB(s)$, where $B(t)$ is the standard Brownian motion and $\Gamma(x)$ is the usual Γ function. The fractional Brownian motion has dependent increments, in essence the fractional Brownian motion displays LRD which is measured by β. Classically the Hurst index, $H \in (\frac{1}{2}, 1)$, indicates the process displays LRD. The parameter β is related to H as $\beta = H - \frac{1}{2}$ (see Beran 1994, p.52–53). If $0 < \beta < \frac{1}{2}$ then the process is said to have LRD, if $-\frac{1}{2} < \beta < 0$ then the process is said to have intermediate–range dependence (IRD) and if $\beta = 0$ then the process is of type (1).

As a particular case of (2), we are interested in the form

$$dX(t) = -\alpha X(t)dt + \sigma dB_\beta(t), \qquad (3)$$

where the $\alpha > 0$ is the drift parameter and the the diffusion function is given by the parameter $\sigma > 0$. This particular expression is chosen because it has a solution (Comte and Renault, 1996) that is equivalent to a stationary process.

Thirty years ago, Black and Scholes (1973) assumed a constant volatility to derive their famous option pricing equation. The implied volatility values obtained from this equation show skewness, suggesting that the assumption of constant volatility is not feasible. In fact, the volatility shows an intermittent behaviour with periods of high values and periods of low values. In addition, the asset volatility cannot be directly observed. The stochastic volatility (SV) model deals with these two facts. Hull and White (1987) amongst others study the logarithm of SV as an Ornstein–Uhlenbeck process. Andersen and Lund (1997) extend the CIR model to associate the spot interest rate with stochastic volatility process through estimating the parameters with the efficient method of moments.

Comte and Renault (1998) specified the fractional stochastic volatility model (FSV), as an extension of the SV with stochastic volatility displaying LRD. This paper proposes a new methodology to estimate the volatility parameters from the returns. Our research shows that the parameter estimation of the FSV model can be transformed to the parameter estimation of a process without LRD. A priori, the assumption of long memory on the stochastic volatility might suggest LRD on the underlier and returns. It is shown that the returns do not display LRD independently of long memory in the volatility values.

1. LRD PROCESSES

In this section we are interested in a fractional stochastic process of type (3). As it can be found in Comte and Renault (1996), these processes have a solution of type,

$$X(t) = \int_0^t A(t-s)dB(s) \quad (4)$$

with $A(x) = \frac{\sigma}{\Gamma(1+\beta)}\left(x^\beta - \alpha\int_0^x e^{-\alpha(x-u)}u^\beta du\right)$. It follows from equation (4) that $X(t)$ belongs to a family of non–stationary Gaussian processes. It is known that an asymptotically equivalent process $\tilde{X}(t)$ can be found,

$$\tilde{X}(t) = \int_{-\infty}^t A(t-s)dB(s), \quad (5)$$

which is still stationary Gaussian with mean zero. The spectral density of this process is defined by,

$$\phi_{\tilde{X}}(\omega) = \phi_{\tilde{X}}(\omega,\theta) = \frac{\sigma^2}{\Gamma^2(1+\beta)}\frac{1}{|\omega|^{2\beta}}\frac{1}{\omega^2+\alpha^2}, \quad (6)$$

which is well–defined for all values $\omega \in \Re$. Thus for values of $\beta \in (0, \frac{1}{2})$, the spectral density behaves as a usual LRD spectral density: decreasing to zero as $|\omega| \to \infty$ and increasing to ∞ as $|\omega| \to 0$. For values of $\beta \in (-\frac{1}{2}, 0)$ the spectral density, $\phi(\omega,\theta)$, decreases to zero as $|\omega| \to \infty$ and $|\omega| \to 0$ and has the maximum at $\omega = \alpha\sqrt{\frac{-\beta}{1+\beta}}$.

The autocovariance function and the spectral density are closely related to each other. The autocovariance function is defined in the time domain while the spectral density is defined in the frequency domain. In summary, the spectral density is the Fourier transform of the autocovariance function and therefore we can derive one from the other. The autocovariance function of $\tilde{X}$ is the inverse Fourier transform of (6),

$$\begin{aligned} \gamma_{\tilde{X}}(\tau) = & \operatorname{cov}(\tilde{X}(t+\tau), \tilde{X}(t)) \\ = & 2\int_0^\infty \phi_{\tilde{X}}(\omega,\theta)\cos(\omega\tau)d\omega. \end{aligned} \quad (7)$$

This explicitly implies,

$$\begin{aligned} \gamma_{\tilde{X}}(\tau) = & \tfrac{2\sigma^2}{\Gamma^2(1+\beta)}\left\{\tfrac{\pi\cosh(\alpha(\tau))}{2\alpha^{1+2\beta}\cos(\beta\pi)} - \right. \\ & |\tau|^{1+2\beta}\Gamma(-1-2\beta) \\ & \left. {}_2F_3[1, 1+\beta, \tfrac{3}{2}+\beta, \tfrac{\alpha^2(\tau)^2}{4}]\sin(\beta\pi)\right\} \end{aligned} \quad (8)$$

with ${}_2F_3$ the generalised hypergeometric function (Prudnikov et al.,1986). Then the variance of $\tilde{X}(t)$ is given by

$$\sigma^2_{\tilde{X}} = \gamma_{\tilde{X}}(0) = \frac{\sigma^2\pi}{\Gamma^2(1+\beta)\alpha^{1+2\beta}\cos(\beta\pi)}. \quad (9)$$

Note that $\sigma^2_{\tilde{X}}$ contains all information of process (5). Gao, Anh and Heyde (2002) and Gao (2004) propose using a continuous–time version of the Whittle contrast function to estimate α, β and σ. What financial data can be modelled with $\tilde{X}(t)$? Indexes such as the S&P 500 are perfect candidates. Figure 1 shows that the S&P 500 is not stationary. In other hand, the returns displayed in Figure 2 are likely stationary, although they are non–Gaussian.

Figure 1 shows the curve of the index from January 1950 until July 2005.

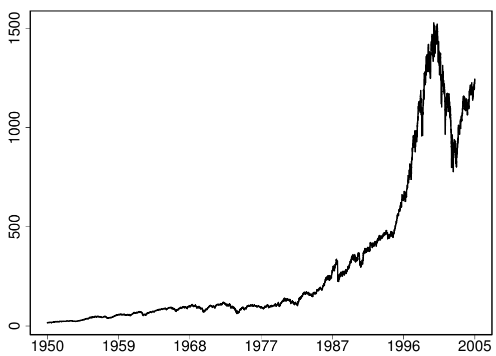

Figure 1: S&P500 daily index from Jan. 1950 until Jul. 2005

The condition of Gaussianity is very restrictive. To the extend of our knowledge, there is no financial data with LRD that can be modelled by $\tilde{X}(t)$ directly.

2. FRACTIONAL STOCHASTIC VOLATILITY

The SV model consists of two stochastic differential equations (SDE) which simultaneously model the price and its volatility. Comte and Renault (1998) extend the SV model to accommodate data whose volatility displays LRD, specifiying the FSV model. The form

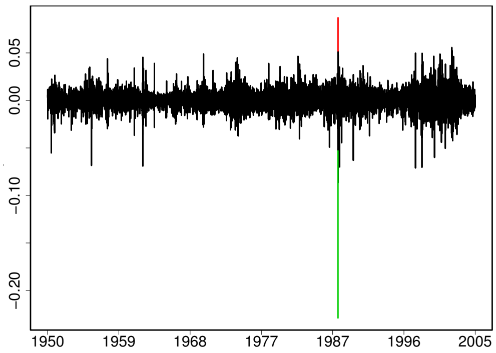

Figure 2: S&P500 returns from Jan. 1950 until Jul. 2005

of interest to this paper is given by,

$$dY(t) = \tilde{v}(t)dB^I(t), \tag{10}$$

$$d\tilde{X}(t) = -\alpha\tilde{X}(t)dt + \sigma dB^{II}_\beta(t), \tag{11}$$

where $\tilde{X}(t) = \ln(\tilde{v}(t))$ and $Y(t) = \ln(S(t))$. The independency between $B^I(t)$ and $B^{II}_\beta(t)$ is assumed.

Equation (10) models the behaviour of the price process $S(t)$ – for instance the value of the S&P 500 at time t. In this equation $\tilde{v}(t)$ is the volatility, a stochastic process modelled by equation (11). The aim of this paper is to find a methodology to estimate α, β and σ.

Equation (11) is a fractional stochastic process of type (3). Thus, $\tilde{X}(t)$ is a Gaussian process with variance $\sigma^2_{\tilde{X}}$ as given in equation (9). Then, $\tilde{v}(t)$ is a log-nomal process with mean $\mu_{\tilde{v}} = \exp(\sigma^2_{\tilde{X}}/2)$, variance $\sigma^2_{\tilde{v}} = \left(\exp(\sigma^2_{\tilde{X}}) - 1\right)\exp(\sigma^2_{\tilde{X}})$ and covariance matrix $K_{\tilde{v}}$ given by,

$$\exp\{\sigma^2_{\tilde{X}}\}\begin{pmatrix} \exp\{\sigma^2_{\tilde{X}}\} - 1 & \dots & \exp\{\gamma_{\tilde{X}}(n\tau)\} - 1 \\ \exp\{\gamma_{\tilde{X}}(\tau)\} - 1 & \dots & \exp\{\gamma_{\tilde{X}}((n-1)\tau)\} - 1 \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \exp\{\gamma_{\tilde{X}}(n\tau)\} - 1 & \dots & \exp\{\sigma^2_{\tilde{X}}\} - 1 \end{pmatrix}$$

Note that $\sigma^2_{\tilde{v}}$ is an expression containing the three parameters of the volatility process.

The solution of (10),

$$Y(t) = Y(0) + \int_0^t \tilde{v}(r)dB^I(r),$$

is non–stationary and non–Gaussian, with mean $Y(0)$. The initial value $Y(0)$ can be assumed to be zero without loss of generality. The covariance function of $Y(t)$ is of the form

$$\begin{aligned}\gamma_Y(\tau) &= \int_0^{\min(t,t+\tau)} E[\tilde{v}^2(r)]dr \\ &= \min(t, t+\tau)\sigma^2_{\tilde{v}}.\end{aligned}$$

Thus the covariance depends on t. As expected $\ln(S(t))$ is non–stationary. However, the returns defined as

$$r_t = \ln\left(\frac{S(t)}{S(t-1)}\right) \text{ for } t = 2 \dots T.,$$

are stationary. This motivates the study of the increments of $Y(t)$,

$$\Delta Y(t, \Delta t) = Y(t+\Delta t) - Y(t). \tag{12}$$

In the continuous case $t, \Delta t \in \Re$ and $\Delta t \to 0$. For the discrete case $t, \Delta t \in \{1, \dots, N\}$ where N is the length of $Y(t)$. The first difference transformation is defined for $\Delta t = 1$. A general formula of the autocovariance function of (13),i.e., $E[\Delta Y(t+\tau, \Delta t_1)\Delta Y(t, \Delta t_2)]$ is given by,

$$\int_{-\infty}^{\infty} e^{i\tau\omega}(1 - e^{i\omega\Delta t_1})(1 - e^{-i\omega\Delta t_2})\frac{1+\omega^2}{\omega^2}dF(\omega),$$

which is independent of t for all $\Delta t_1, \Delta t_2$ and $\tau \in \Re$. If $\Delta Y(t, \Delta t) \in L^2(\Re)$ then, $Y(t)$ is said to be wide-sense stationary or second-order stationary (see Doob 1953, Yaglom 1987). When $F(\omega)$ is absolutely continuous, we can find the spectral density $\phi_{\Delta Y}(\omega)$ as the derivative of F. If we assume that the increments are of the same size, i.e. $\Delta t_1 = \Delta t_2 = \Delta t$, then the form of the autocovariance is,

$$\gamma_{\Delta Y}(\tau) = \begin{cases} \sigma^2_{\tilde{v}}(\Delta t - |\tau|) & |\tau| < \Delta t \\ 0 & |\tau| \geq \Delta t \end{cases} \tag{13}$$

which Fourier transform provides the spectral density,

$$\phi_{\Delta Y}(\omega, \theta) = \frac{\sigma^2_{\tilde{v}}}{2\pi}\left(\frac{\sin(\omega\Delta t/2)}{\omega/2}\right)^2. \tag{14}$$

When $\omega \to 0$, the spectral density $\phi_{\Delta Y}(\omega, \theta)$ is proportional to $(\Delta t)^2$. The spectral density goes to zero when $\omega \to \infty$. As it can be seen in Figure 3, this is the spectral density of a process without LRD as does not explode at $\omega = 0$. In addition, the autocorrelation function (13) decreases to zero. We conclude that the series of equal size increments of $Y(t)$ does not display LRD.

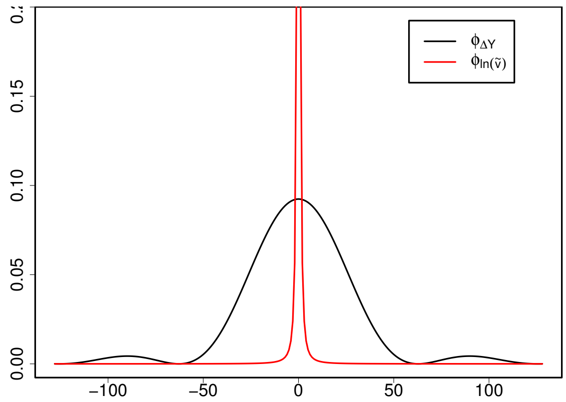

Figure 3: Comparison of spectral density of a IRD and LRD process

Empirically, the returns of the S&P 500, Dow Jones, Nasdaq, Nikkei, etc. do not display LRD. A priori, one might expect that the LRD of the volatility process would be transpassed into the underlying process. Our study shows, the LRD of the volatility may not be sufficient to ensure the LRD of the returns.

3. WHITTLE ESTIMATION

Some detail discussion on spectral analysis involving short–range dependent stationary time series can be found in §10 of Brockwell and Davis (1991) and Priestly (1981). Gao, Anh and Heyde (2002) propose a continuous–time periodogram of the form

$$I_Y^N(\omega) = \frac{1}{2\pi N} \left| \int_0^N e^{-i\omega t} Y(t) dt \right|^2,$$

where $N > 0$ is the upper bound of the interval $[0, N]$, on which each $Y(t)$ is observed.

As in Gao (2004), this paper uses an extended continuous–time version of the discrete Gauss–Whittle contrast function used by Dahlahaus (1989). For processes with LRD, we need to get a weight function invloved to ensure that the contrast function is well–defined. This is not the case for processes with IRD, thus the contrast function used in this paper is,

$$L_Y^N(\theta) = \frac{1}{4\pi} \int_0^\infty \left\{ \log(\phi_Y(\omega, \theta)) + \frac{I_Y^N(\omega)}{\phi_Y(\omega, \theta)} \right\} d\omega.$$

The minimum contrast estimator of θ is defined as

$$\bar{\theta} = \arg\min_{\theta \in \Theta_0} L_Y^N(\theta),$$

where Θ_0 is a compact subset of Θ.

As can be seen from Theorem 3.1 of Gao (2004), both the convergence in probability and the asymptotic normality of $\bar{\theta}$ hold automatically for the case where $\theta \in \Theta_1 = \{\theta = (\alpha, \beta, \sigma) : \alpha > 0, 0 < \beta < \frac{1}{2}, \sigma > 0\}$. As in Casas and Gao (2004), the convergence also holds for $-\frac{1}{2} < \beta \leq 0$ which is of our concern in this paper.

4. CONCLUSIONS

The estimation of the volatility process is one of the most difficult problems in econometrics. Neither volatility simulation techniques nor volatility data collection are completely satisfactory. Instead, the authors propose a technique to estimate the volatility from the returns. The main assumption is that the volatility process is Gaussian with the property of LRD. Therefore the probability distribution of the returns is non–Gaussian. Suprisingly, the returns do not inherit the LRD property from the volatility process. The main contribution of this paper is the simplication of an estimation problem for process with LRD to an estimation problem for processes without LRD.

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