Modelling Velocity Structures In Turbulent Floods Using Proper Orthogonal Decomposition

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EXTENDED ABSTRACT

In New Zealand, there are over 100 communities that are flood prone therefore require good models to predict their vulnerability and also the likely flood depths. Previous work has shown that numerical models under predict water levels in the centre of the flow. One of the causes is the wave action that is prevalent in floodwater flows. Therefore it is important to develop a model of wave action.

The approach taken was to develop a model that included the underlying velocity field associated with the waves. To do this the turbulent structures needed to be understood. This meant that the analysis may also provide data about the interaction of the water flow and ground surface and therefore the flow resistance.

There are various techniques available to model the turbulence structure. Turbulence of incompressible water flow (without sediment) can be modelled completely using the Navier-Stokes equations. However for even a small part of a river channel, the size of the problem is many orders of magnitude larger than even the world's largest super-computers can handle. This means that other techniques are necessary to describe the turbulence. One of these is Proper Orthogonal Decomposition (POD).

POD provides a way of decomposing the velocity vectors into modes of different scales in each direction similar to a Fourier series. Therefore it requires detailed data. Particle Image Velocity (PIV) is an ideal method to obtain such data.

An undular hydraulic jumps was chosen as an ideal structure to model as it is a strong wave in water flow and similar to the rooster tail typical of New Zealand rivers shown in figure 1. In this paper, we explain the application of POD to PIV image data of a slice through an undular hydraulic jump or type of standing wave. The data is two-dimensions, in the direction of flow and the vertical.

This is the first time PIV flow data from an undular hydraulic jump has been analysed using POD. We identify flow structures and develop covariance functions from velocity correlations across this random flow field. We discuss the structures within the context of this data and their significance.

This will lead to the development of a model based on the Navier-Stokes equations using the covariance functions to simulate the turbulent fluctuations. As there are a wide range of structures within the hydraulic jump a wide variety of flow situations can be modelled. Improvements could be made by analysis of further PIV data especially from higher Reynolds Numbers.

Figure 1. A New Zealand River (Opihi River, South Canterbury) in flood showing rooster tail standing waves.

The model can therefore be run for the various types of situations in a river or a flood plain to predict wave heights which can be used with the results from a hydrodynamic model such as Hydro2de or Mike21. In addition the flow resistance values for these areas may also be able to be calculated to use in a hydrodynamic model. These results can be put into a database of a local authority where it can be used to give building floor levels in flood plains.

1. INTRODUCTION

Presently in New Zealand flood modelling is widely undertaken on most of the major rivers and flood plains. On the river control systems, these models are used to estimate their flood carrying capacity while on the flood plains they are used to model the flooding from super-design floods with maps being produced showing the likely flood extent and inundation depths. This information is used by local authorities for new buildings in these areas. Floor levels of new buildings in New Zealand are required to be above flood levels that vary between the 50 year return period flood event, to the 500 year flood event. A typical flood map is shown in Figure 2.

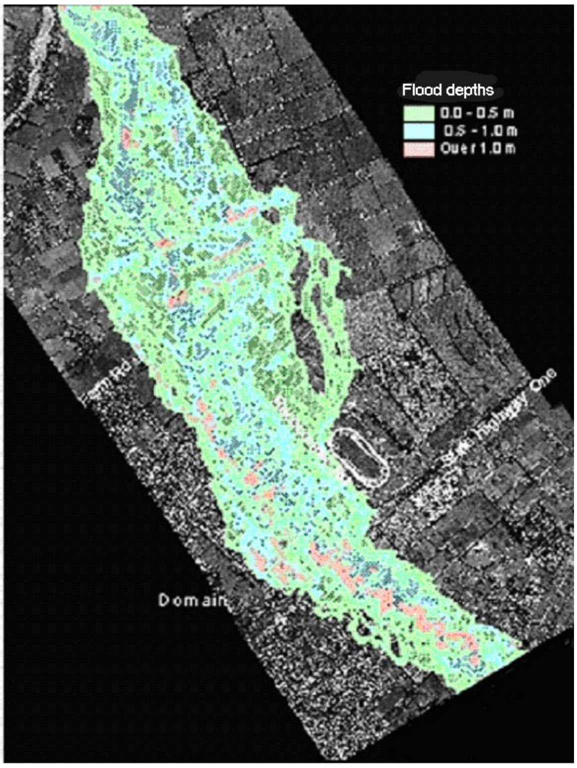

Figure 2. Map showing extent of flood inundation for the estimated 200 year return period event for Ashburton, New Zealand (Modelled using Hydro2de with a 10 m grid of 370,000 points).

The information contained in these maps needs to have all the factors of the natural system included especially those that are not included in the modelling. Previous work by Connell (2001) on the Waihao River flood plain on the Canterbury Plains in New Zealand showed that these models underpredict the flood levels on the flood plain in the centre of the flow. The reasons for this are surmised to be wave action and local afflux or heading of water up against structures within the flood plain where the flood levels were measured.

The effects of afflux against structures can be assessed by including more detail in the models e.g. increasing the resolution of the model to include the buildings. However the effects of wave action cannot be included in the present type of models commonly used for this work (Two-dimensional hydrodynamic models.) These effects will be greater in a river channel where standing waves such as those shown in Figure 1 can occur.

The approach chosen was to develop a model of wave action that included the underlying turbulent flow structure. This also meant that the model could possibly have the ability to assess the flow resistance. This is very important as resistance factors for the hydrodynamic models are still only estimated with information from river gaugings and empirical techniques. Use of photographs with gauged data to calculate the resistance factor is still a well used technique e.g. (Hicks and Mason 1991). This publication also shows how the standard resistance factors, in this case Manning's 'n' and the Chezy 'C' change significantly with depth for a given reach.

Models have been developed to solve the Navier-Stokes equations in water (without sediment) and can provide a complete solution. However to model the complete range of flow scales, even for a small area of river flow, requires a model many orders of magnitude greater than the largest supercomputer available. Therefore techniques are required to reduce the size of the problem minimising the effects of the loss of detail.

This paper presents the beginnings of developing a model. One method of assessing the flow structure within the flow is called Proper Orthogonal Decomposition (POD). This technique was first used to assess the flow structure by Bakewell and Lumley (1967). Since, this technique has been developed mainly in the aeronautical field e.g. Holmes *et al.* (1996). To develop such models analysis of the data structures of flow with surface waves in this cases the undular hydraulic jump, needs to be undertaken. The best technique that collects data in a form to be analysed by POD is Particle Image Velocimetry (PIV).

2. WAVE STUDIES

The study first examined types of waves in shallow water flow. A wave typical of Canterbury Rivers, are rooster tails that occur in flow close to critical. Such an example is shown in Figure 1. Much initial work was undertaken by Kennedy (1961).

Subsequent analysis lead to data by Yuen and Kennedy (1971) of flow over wavy beds was analysed by Hosoda and Minamimoto (2002) using the Boussinesq equations. The undular (or wave) hydraulic jump has also been analysed using the Boussinesq equations by Peregrine (1966) and Frazao and Zech (2002). This jump is very similar to a rooster tail and is in effect a very wide rooster tail as shown by Figure 3 (courtesy of H. Chanson's web site, University of Queensland).

Figure 3. An undular Hydraulic Jump (the river bore case).

The similarity between these two hydraulic phenomena meant that data on the undular hydraulic jump would be a good starting point upon which to develop a model of wave action in rivers.

3. PARTICLE IMAGE VELOCIMETRY (PIV) DATA

As stated above, a detailed velocity field data is needed to undertake a POD analysis. PIV data is ideal for this. PIV collects a detailed velocity field within the water flow with data points less than 1 mm apart. This is done by inserting neutrally buoyant silver coated glass spheres into the flow and then taking pairs of photographs very close together, 75 μs for these data, with the particles being highlighted by a laser sheet pulse sent at the same time as each photograph. At this time spacing the particles have not moved greatly and therefore the water movement can be measured by assuming that the sphere's closest to each other from each pair of photographs is the same sphere. Cross-correlation software has been developed to undertaken this analysis.

Data from a laboratory undular hydraulic jump experiment by Lennon (1994) has been used to develop the model. The Reynolds number of the flow was about 24,000, with in super-critical inflow having a depth of 0.0314 m and velocity of 0.762 m/s and Froude number of 1.37. The downstream Froude number averaged about 0.75 and the depth averaged 0.047 m varying about these values along the undular waves downstream.

The data obtained was spaced on a grid size of 0.085 mm. The data set consisted of 11 overlapping views (except where there was a metal plate in the flume between two views) of the jump each with over 5000 data points. A typical velocity field from a view is shown in figure 4. At each view point 400 photograph pairs were taken. This enabled a good set of statistics to be obtained for each point over the whole velocity field.

The data is of the form u(x,y) and v(x,y) where u and v are the x and y velocities respectively and x and y are the x and y positions on the domain. The time step between each pair of photographs was about 1 second which meant that the data between time steps cannot show the development of the flow structures as the flow has changed too much between each photograph. However this was not an issue as the method generated covariance functions from the data's the flow statistics of the data sets.

4. PROPER ORTHOGONAL DECOMPOSITION (POD)

Proper orthogonal decomposition is known by several other names including Karhunen-Loeve (KL) expansion, principal components analysis and is very similar to singular value decomposition. The following is a brief one-dimensional description, similar to Chambers et al. (1988), to give an overview of the method as the PIV data is two-dimensional. For more details see Berkooz et al. (1993) and Bakewell and Lumley (1967). For a two-dimensional description see Liu et al. (2001).

The POD is a generalised Fourier expansion of a random field, in this case the velocity u(x), using the sum of orthogonal basis functions, φ_n ,

$$u(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \qquad (1)$$

with random Fourier coefficients,

$$a_n = \int_0^h u(x)\varphi_n(x)dx . \qquad (2)$$

For orthogonality,

$$\int_0^h \varphi_n(x)\varphi_m(x)dx = \partial_{mn} \qquad (3)$$

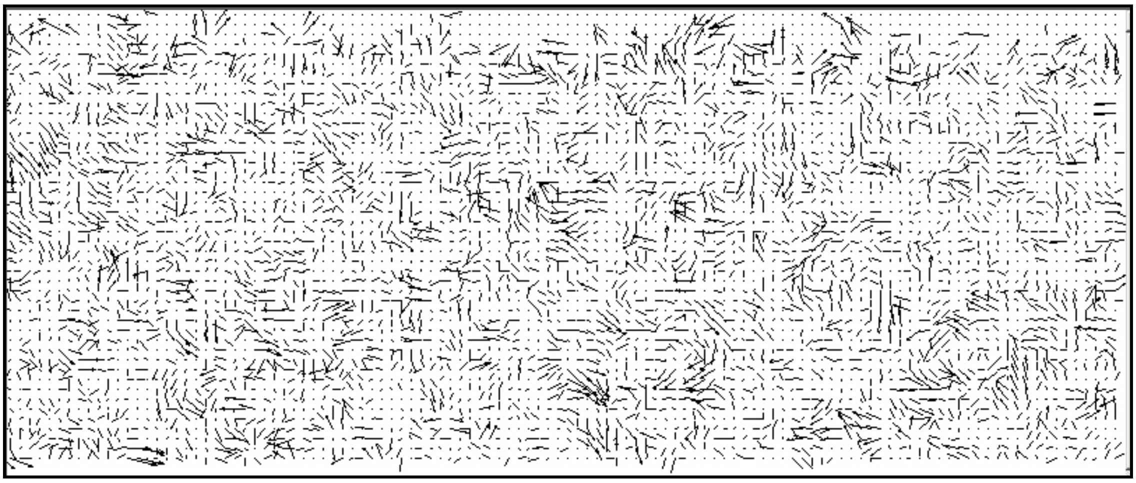

Figure 4. Part of a velocity field obtained from a PIV analysis. Note that the mean velocities (for each horizontal line of data points) have been removed from the plot to show the flow structures. Note that the top 10 lines are above the water surface and appear in the data due to reflections from the water surface.

where ∂_{nm} is the Kronecker delta.

To minimise the mean square error from a partial sum of N terms (compared to the infinite number in equation (1)), leads to a homogeneous Fredholm equation for the basis functions,

$$\int_0^h R(x,x')\varphi_n(x')dx' = \lambda_n \varphi_n(x) \qquad (4)$$

where $R(x,x') = \langle u(x)u(x')\rangle$ the correlation function that can be derived from the PIV velocity field and φ_n and λ_n are the eigenfunctions and eigenvalues. By Mercer's theorem,

$$R(x,x') = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x)\varphi_n(x') \qquad (5)$$

from which can be shown that random Fourier coefficients are orthogonal variables with mean square values equal to the eigenvalues λ_n,

$$\langle a_m a_n \rangle = \lambda_m \partial_{nm} \qquad (6).$$

The integral equations can be solved by reducing them to matrix eigenvalue problems using the trapezoidal quadrature. Then equation (4) becomes in matrix form (Press *et. al.* 1986),

$$\mathbf{R \cdot f} = \lambda\, \mathbf{f} \qquad (7)$$

The Fredholm equation is a homogeneous one of the second kind which means that the kernel may have lost its symmetry with the quadrature. However the symmetry can be restored by using a diagonal matrix of the weights used in the quadrature i.e. D = diag (w_j), where w_j are the weights giving,

$$\mathbf{R \cdot D \cdot f} = \lambda\, \mathbf{f} \qquad (8)$$

Multiplying by $D^{1/2}$ gives,

$$\mathbf{D^{1/2} \cdot R \cdot D^{1/2} \cdot h} = \lambda\, \mathbf{h}, \qquad (9)$$

which is now a symmetric eigenvalue problem. As the kernel is square integrable these eigenvalues will provide a good approximation for the lowest N eigenvalues of the integral equation.

The Fredholm equation can also be solved analytically in many cases.

5. ANALYSIS

5.1. Two Point correlation functions - Covariance Structure

Analysis of the covariance structure of the velocity field was undertaken by developing software within the Matlab program.

In each field two-dimensional two point velocity correlations were undertaken. The correlations were undertaken in both the flow direction and vertical direction. This was undertaken from several chosen depths in the flow. The data at each depth was averaged over the ensemble of 100 of the data sets for each field and the velocity products with the chosen depths of flow were calculated and divided by the variances to obtain the correlations. There were points where data was missing and these were eliminated from the calculation. The results are shown for the first view or velocity field in figure 5.

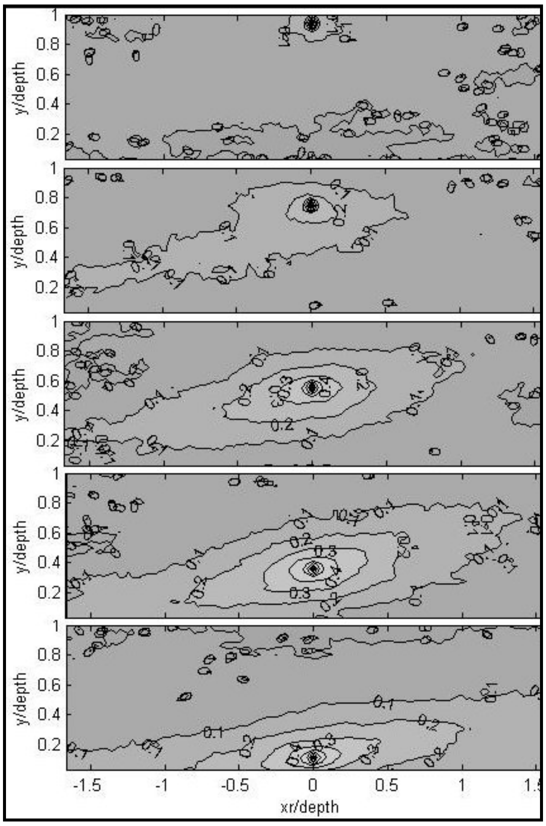

Figure 5. X-velocity correlation plots at 5 depths in the flow (y = 0.11h, 0.36h, 0.55h, 0.75h and 0.94h) for the first of the 11 fields.

As can be seen from Figure 5 show the x-velocities of the flow are strongly correlated over a considerable distance upstream and downstream at an angle of about 8 degrees with the bed. This angle is probably due to the flow structures arising from the bed. These structures are known as sweeps and bursts. They have a strong effect on the average velocity in the field as it varied between data sets or photograph pairs from 0.7 m/s to 0.84 m/s (a 20 % range) for a length of flow that was 3 times is depth.

However the plots in figure 5 also show that in the top 10 % of the flow the x-velocity correlation distance is small and is similar to the y-velocity correlations distances shown in figure 6 below.

The correlation distances shown in figure 6 are not great and this reflects the size of the structures in the vertical direction. The average y-velocity did not change significantly from data set to data set whereas the x-velocity did varying by up to 20%. This meant when the ensemble averaging took place the x-velocity correlation distances were much larger than if just sample averaging was used.

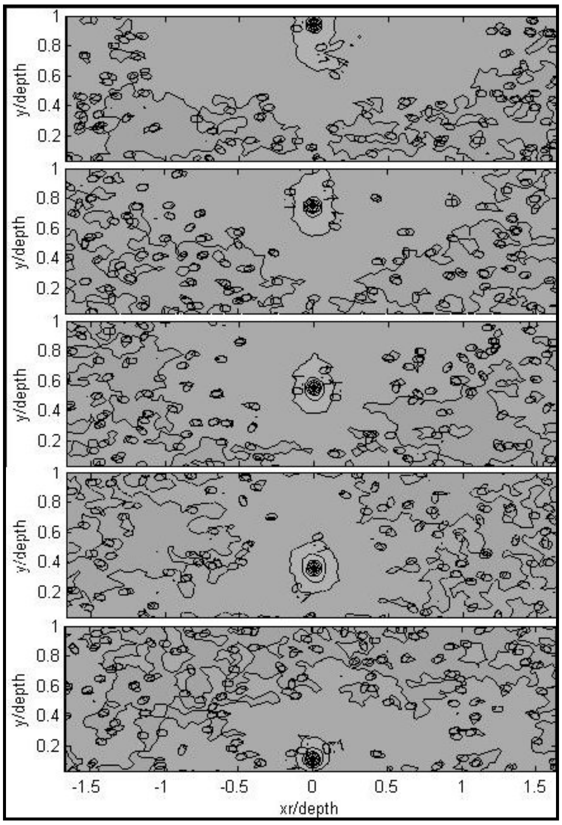

Figure 6. Y-velocity correlation plots at 5 depths in the flow (y = 0.11h, 0.36h, 0.55h, 0.75h and 0.94h) for the first of the 11 fields.

These correlations are very similar to those calculated by (Liu *et. al.* 2001) for this flow between two plates with a similar Reynolds number.

5.2. Eigenfunctions

The eigenfunctions are solved both numerically and if possible analytically.

6. CONCLUSIONS

The analysis shows that PIV data for uniform flow within an undular jump has statistical covariance properties that vary across the flow. Therefore the flow can be classed as a two-dimensional second order random field. Analysis of the structures under the wave is not able to be presented here due to space considerations. With the covariance function calculated from the analysis there are several options available to build a model. The covariance structures can be inserted into a model based on the two-dimensional Navier-stokes equations (the same two directions as the data, i.e.

the vertical and flow directions. Another option is use a fully stochastic process such as the Ornstein-Uhlenbeck process.

A model so developed will model the undular jump well but not other flows with different Reynolds numbers and Froude numbers. This will mean that the model will be limited in the use that it is intended for which is a wide variety of flow situations in rivers or on flood plains.

Further data or quite possibly more detailed analysis of this data will be necessary to develop a generic model that can be used in any flow situation, including considerably different Reynolds and Froude numbers.

7. ACKNOWLEDGMENTS

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