Smooth estimation of yield curves by Laguerre functions

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ABSTRACT

This paper is concerned with the estimation of the nominal yield curve by means of two complementary approaches. One approach models the yield curve directly while the other focuses on a model of the forward rate from which a description of the yield curve may be developed by integration of the forward rate specification. This latter approach may be broadly interpreted as a generalisation of the widely used parametric functional form proposed by Nelson and Siegel (1987) for yield curve estimation. Nelson and Siegel describe the yield curve in terms of three linear factors commonly referred to as the level factor, the slope factor and the curvature factor, together with a nonlinear factor, say λ, which represents a time-scale. In a recent paper, Diebold and Li (2005) use the Nelson-Siegel method to fit the yield curve for US bonds. In their application, the value of λ (the parameter representing the time-scale) is fixed, leaving only the level, slope and curvature factors with which to capture the behaviour of US yields. Although this approach has the advantage of simplicity in terms of implementation, it is likely to be suboptimal if the fit of the yield curve is sensitive to the choice of the time-scale parameter.

In this article, a limited empirical calibration exercise is performed using gilts yield curves published by the Bank of England. The data set comprises weekly yield curves for maturities from 1 to 19 years at 6-monthly intervals for the time period January 1985 to December 2004. In this exercise, the 3-factor Nelson-Siegel form with λ assigned the value suggested by Diebold and Li on theoretical grounds proved inadequate to capture the variation in the shape of the UK yield curve. In fact, there was no value of λ for which the Nelson-Siegel specification provided an adequate fit to the shape of the UK yield curve, a finding which suggests that this model may not be flexible enough to describe commonly occurring patterns in observed yields.

This article presents two possible ways in which the Nelson-Siegel model for fitting yield curves may be generalised. These generalisations are based on the Fourier-Laguerre representation of a continuous function in $(0, \infty)$, and enjoy the advantage that they retain the overall structure of the Nelson-Siegel model and its ease of implementation. Both models describe the yield curve as a sum of linear factors multiplying nonlinear functions of maturity based on Laguerre functions, all of which use a common value for λ. Just as for the Nelson-Siegel model of yields, these generalised models may be implemented within a simple least-squares framework.

Both the yield-based and forward rate based models are applied to the UK Gilts data for various numbers of factors. Both 3-factor models exhibit the same poor quality of fit as that experienced by the Nelson-Siegel model. Both variants of the generalised 4-factor model performed equally well when fitted to UK yield curve, and both proved superior to the fit of the Nelson-Siegel model. Rather interestingly, the move to five factors did not significantly improve the quality of the fit.

A matter of concern when using the Nelson-Siegel model to fit yield curves is the sensitivity of the fit to the value of the time-scale parameter λ. The value of λ used in the Diebold and Li (2005) investigation of the yield curve for US bonds was unsuitable for the UK yield curve. In particular, the sensitivity of the Nelson-Siegel model to the value of λ was such that the factors of the model could not compensate for an inappropriate choice of λ. On the other hand, the improved fit to the UK yield curve achieved by the generalised models with four factors is, in addition, achieved with less sensitivity to the value of λ. This sensitivity to the value of λ is further reduced with five factors although, as commented previously, the quality of the fit is not improved significantly.

Finally, although the optimal fit achieved by the generalised model of yields is indistinguishable from that based on the forward rate, the sensitivity of this fit to the value of λ is greater for the former than the latter. It is conjectured that this difference stems from the different behaviour of both models at long maturities, and suggests that the model based on forward rates is to be preferred.

1 INTRODUCTION

The yield curve is the plot of the yield to maturity of zero-coupon bonds against maturity. In practice the yield curve is not observed but must be extracted from observed bond prices for a set of (usually) incomplete maturities. One approach to yield construction is due to Fama and Bliss (1987) who develop a piecewise constant representation of forward rates from observed bond prices. These forward rates are then used to generate a smooth yield curve in one of two ways. The first fits a parametric functional form to the unsmoothed forward rates and a smooth yield curve is then computed. Alternatively, the unsmoothed forward rates are averaged to obtain the so-called unsmoothed Fama-Bliss yields to which yield curve with prescribed parametric functional form is then fitted.

One of the most widely used parametric functional forms used in yield curve estimation was proposed by Nelson and Siegel (1987). Essentially this form describes the yield curve in terms of three factors, which may be broadly interpreted as the level, slope and curvature of the yield curve, and one further factor that represents a time scale. In a recent paper, Diebold and Li (2005) use the Nelson-Siegel method to fit the yield curve for US bonds. In their application, the value of the parameter representing the time scale is fixed, leaving only three linear factors with which to capture the behaviour of yields. Although this approach has the advantage of simplicity in terms of implementation, it is likely to be suboptimal if the fit of the yield curve is sensitive to the choice of the time scale parameter.

The main contribution of this paper is to propose a general framework within which to represent either the yield curve or the forward rate of interest. This approach is based on the Fourier-Laguerre representation theorem of functions defined on $(0,\infty)$, and enjoys the crucial advantage that it allows an arbitrary number of independent linear factors to be introduced into the model of either the yield curve or the forward rate. It will be argued that these additional factors can play an important role in reducing the sensitivity of the fit of the yield curve to the choice of the time-scale parameter. The simplicity of the Diebold and Li (2005) approach is thus maintained, while none of the theoretical properties of the Nelson-Siegel functional form are compromised.

A limited empirical calibration exercise is performed using gilts yield curves published by the Bank of England. The data set comprises weekly yield curves for maturities from 1 to 19 years at 6-monthly intervals for the time period January 1985 to December 2004. The 3-factor Nelson-Siegel form with the time-scale parameter fixed proved inadequate to capture the variation in the shape of the UK yield curve. The generalised 4-factor variant introduced in this paper was superior in terms of fit. However, the addition of further factors did not significantly improve the quality of the fit, but instead reduced the sensitivity of the fit to the choice of time scale.

The paper is structured as follows. Section 2 outlines the Nelson-Siegel functional form and a generalisation to this model proposed by Svensson (1995). The main theoretical contribution of this paper is contained in Section 3 and the subsections thereof. The Fourier-Laguerre representation is developed directly for the yield curve and also for the forward rate. Section 4 applies the Fourier-Laguerre approach to UK Gilts data. A brief conclusion and directions for future research are given in Section 5.

2 THE NELSON-SIEGEL MODEL

Let t be current time and let $f(u)$ denote the forward interest rate at time $(t+u)$, then the current price of a zero-coupon bond maturing at par at time $(t+\tau)$ is

$$P(\tau) = \exp\Big[-\int_0^\tau f(u)\,du\Big]. \tag{1}$$

The yield of the bond, $y(\tau)$, the price of the bond, $P(\tau)$, and the forward rate, $f(u)$, satisfy

$$y(\tau) = -\frac{\log P(\tau)}{\tau} = \frac{1}{\tau}\int_0^\tau f(u)\,du. \tag{2}$$

The classical term-structure problem requires the estimation of the smooth yield curve $y(\tau)$ from observed bond prices. In recent years the method of choice has been to compute the implicit forward rates required to price successively longer maturity bonds at the observed maturities. These are called unsmoothed forward rates. The smoothed forward rate curve is then obtained by fitting a parametric functional form to these unsmoothed rates. One common choice proposed by Nelson and Siegel (1987) is

$$f(u) = \beta_1 + \beta_2 e^{-\lambda u} + \beta_3 \lambda u e^{-\lambda u}. \tag{3}$$

Alternatively, the unsmoothed forward rates can be converted into unsmoothed yields by interpreting equation (2) as an averaging procedure. By choice of the parameters β_1, β_2, β_3 and λ, these unsmoothed yields are now fitted to the functional form

$$\begin{aligned} y(\tau) &= \beta_1 + \beta_2\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) \\ &\quad + \beta_3\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\Big) \end{aligned} \tag{4}$$

which is the yield curve corresponding to the Nelson and Siegel forward rate (3). The constant term β_1 in this model is interpreted as the *level* of the yield curve, that is, it is the limiting value of the yield curve $y(\tau)$ as $\tau \to \infty$. Therefore the parameter β_1 is required to be positive. The term $\big(1 - e^{-\lambda\tau}\big)/(\lambda\tau)$ is interpreted as the *slope* of the yield curve. This function decreases monotonically with maturity. Since $y(\tau) \to \beta_1+\beta_2$ as $\tau \to 0^+$, then the instantaneous rate depends on both the level and the slope of the yield curve. The term $\big(1 - e^{-\lambda\tau}\big)/(\lambda\tau) - e^{-\lambda\tau}$ is related to the *curvature* of the yield curve. The motivation for this claim is based, first, on the observation that this function rises with maturity and then falls asymptotically to zero as maturity increases, and second, on the observation that this function is essentially the gradient of the slope function $\big(1-e^{-\lambda\tau}\big)/(\lambda\tau)$ in view of the identity

$$\Big(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\Big) = -\tau\frac{d}{d\tau}\Big(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\Big).$$

In a recent paper, Diebold and Li (2005) argue that the parameters β_1, β_2 and β_3 denoting respectively the level, slope and curvature of the yield curve may be treated as dynamic factors. Time series of these factors are obtained easily by estimating the Nelson-Siegel model (4) from unsmoothed yields data for all available bond maturities on a given date. The time series of observations on the parameters β_1, β_2 and β_3 are generated by repeating this procedure for a sequence of dates. If the factors can be forecast successfully, then bond prices can be predicted successfully. Moreover, this strategy would need to be more successful than focussing directly on bond prices, or on forecasting the entire yield curve itself (rather than the loadings on the factors thought to underly the yield curve).

Clearly the efficacy of the Diebold and Li (DL) approach will depend on how accurately the model fits the yield curve at each date, and also on how well the factors may be forecast from their resulting time series. This paper focusses on the former problem and highlights two potential limitations of the DL approach. The first limitation concerns the treatment of the parameter λ. DL set $\lambda = 0.0609$ so that the nonlinear Nelson-Siegel function may be implemented as a linear regression. The motivation for this choice of λ is to ensure that $\big(1-e^{-\lambda\tau}\big)/(\lambda\tau)-e^{-\lambda_1\tau}$ reaches its maximum value at a maturity of 30 months in recognition of the fact that the yields of 2 year and 3 year bonds are generally acknowledged to be important benchmark rates.

The second limitation of the Diebold approach is the restriction on the number of factors. The three factor approach advocated by Nelson and Siegel (1987) has been extended by Svensson (1995) to a 4-factor representation

$$f(u) = \beta_1 + \beta_2 e^{-\lambda_1 u} + \beta_3\lambda_1 u e^{-\lambda_1 u} + \beta_4\lambda_2 u e^{-\lambda_2 u}$$

with corresponding yield specification

$$\begin{aligned} y(\tau) \;\;=\;\; & \beta_1 + \beta_2\Big(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau}\Big) \\ & +\beta_3\Big(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\Big) \\ & +\beta_4\Big(\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\Big). \end{aligned}$$

This specification is problematic. Not only are there now two λ parameters to estimate, but the functional form suffers from the difficulty that it is degenerate when the values of the parameters λ_1 and λ_2 coincide. When λ_1 and λ_2 take different values, the model specification leads to an underlying parameter estimation problem that is no longer straightforward since it now involves multi-dimensional minimisation.

Section 3 now provides an alternative interpretation of the Nelson-Siegel model that can be extended naturally to an arbitrary number of factors. This generalisation can accommodate all possible yield curve shapes, and also enjoys the advantage that the individual factors in the model represent mutually orthogonal contributions to the final specification of the yield curve.

3 THE FOURIER-LAGUERRE APPROACH TO MODEL SPECIFICATION

There are two ways to proceed; either model the yield curve $y(\tau)$ and derive the forward rate from the definition

$$f(\tau) = \frac{d}{d\tau}\big(\tau y(\tau)\big), \tag{5}$$

or follow the Nelson-Siegel approach in which the forward rate is modelled and the yield curve derived from this model using equation (2). While both approaches yield a similar fit, it will become clear later that the method that fits the yields directly is more sensitive to the value of λ than the approach based on a model of the forward rate.

3.1 Preliminaries

The generic form for the function to be modelled in this paper, namely the yield curve or the forward rate, is motivated by a representation theorem which asserts that if $\phi(\tau)$ is a continuous function in $(0, \infty)$ satisfying

$$\int_0^\infty e^\tau\,\big[\,\phi(\tau) - \beta_1\,\big]^2\,d\tau < \infty\,,$$

then $\phi(\tau)$ can be represented pointwise by the Fourier-Laguerre series

$$\phi(\tau) = \beta_1 + e^{-\lambda\tau} \sum_{k=0}^{\infty} c_k L_k(\lambda\tau) \tag{6}$$

where $L_k(x)$ is the Laguerre polynomial of degree k, λ is a positive parameter and $c_0, c_1, \cdots$ are coefficients to be determined (see, for example, Canuto, Hussaini, Quarteroni and Zang, 1988). The specification of $\phi(\tau)$ in equation (6) must be estimated from a finite number of maturities of restricted duration, and therefore in practice it is convenient to represent $\phi(\tau)$ by a sum of $(N+1)$ Laguerre polynomials, say

$$\phi(\tau) = \beta_1 + e^{-\lambda\tau} \sum_{k=0}^{N} c_k L_k(\lambda\tau)\,, \tag{7}$$

where the parameters λ, β_1 and the coefficients $c_0, \ldots, c_{\text{N}}$ are to be estimated from observations of bond prices at various states of maturation.

The case in which the model is written for the yield curve is considered separately from the case in which the forward rate is the subject of the model. Both approaches, however, are based on Laguerre polynomials in which $L_k(x)$, the Laguerre polynomials of degree k, is defined conveniently by Rodrigues formula

$$L_k(x) = \frac{e^x}{k!}\,\frac{d^k}{dx^k}\big(x^k e^{-x}\big)\,. \tag{8}$$

3.2 Model of the yield curve

Suppose that the yield curve is modelled by the parametric form

$$y(\tau) = \beta_1 + e^{-\lambda\tau} \sum_{k=0}^{N} c_k L_k(\lambda\tau)\,, \tag{9}$$

then β_1, λ and the coefficients $c_0, \cdots, c_{\text{N}}$ are chosen to minimise

$$\sum_{j=1}^{M} \Big[\, y(\tau_j) - \beta_1 - e^{-\lambda\tau_j} \sum_{k=0}^{N} c_k L_k(\lambda\tau_j)\,\Big]^2 \tag{10}$$

where $\tau_1, \cdots, \tau_{\text{M}}$ are a series of maturation times for a given class of bond and $y(\tau_1), \cdots, y(\tau_{\text{M}})$ are the associated yields. Given the specification (9) of $y(\tau)$, the associated forward rate of interest is

$$f(\tau) = \beta_1 + \sum_{k=0}^{N} c_k \frac{d}{d\tau}\Big(\tau e^{-\lambda\tau} L_k(\lambda\tau)\Big). \tag{11}$$

in view of equation (5). Taking the Rodrigues formula (8) as the definition of $L_k(x)$, it can be shown that

$$\frac{d}{dx}\big(xe^{-x}L_k(x)\big) = e^{-x}\,\big[(k+1)L_{k+1}(x) - kL_k(x)\big]\,.$$

This result is now used in equation (11) to give

$$f(\tau) = \beta_1 + e^{-\lambda\tau} \sum_{k=1}^{N} k\big(c_{k-1} - c_k\big)\, L_k(\lambda\tau) \tag{12}$$

after re-indexing the summation. To summarise, in this approach the parameters of the yield curve given by the specification (9) are estimated directly from observations of bond prices at various states of maturation, and the associated forward rate is determined from equation (12) with known values for β_1, λ and the coefficients $c_0, \cdots, c_{\text{N}}$.

3.3 Model of the forward rate

Suppose now that the forward rate is modelled by the parametric form

$$f(u) = \beta_1 + e^{-\lambda u} \sum_{k=0}^{N} c_k L_k(\lambda u) \tag{13}$$

in which the parameters β_1, λ and the coefficients $c_0, \cdots, c_{\text{N}}$ are to be estimated from observations of bond prices at various states of maturation. The yield curve is first calculated from the forward rate using equation (2) to obtain

$$\begin{aligned} y(\tau) &= \frac{1}{\tau}\int_0^{\tau} \Big[\beta_1 + e^{-\lambda u} \sum_{k=0}^{N} c_k L_k(\lambda u)\Big]\, du \\ &= \beta_1 + \frac{1}{\lambda\tau}\sum_{k=0}^{N} c_k \int_0^{\lambda\tau} e^{-x}\, L_k(x)\, dx\,. \end{aligned} \tag{14}$$

When $k \geq 1$, it follows directly from Rodrigues formula (8) that

$$\int_0^{\lambda\tau} e^{-x}\, L_k(x)\, dx = \frac{1}{k!}\,\frac{d^{k-1}}{dx^{k-1}}\,\big(x^k e^{-x}\big)\Big|_{x=\lambda\tau}\,.$$

Further progress takes advantage of the identity

$$\frac{1}{k!}\,\frac{d^k}{dx^k}\,\big(x^{k+1}e^{-x}\big) = xe^{-x}\sum_{j=0}^{k} L_j(x)\,,$$

which in combination with equation (14) now leads to the formula

$$\begin{aligned} y(\tau) \;\; &= \;\; \beta_1 + c_0\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) \\ &\qquad + e^{-\lambda\tau}\sum_{k=1}^{N}\frac{c_k}{k}\sum_{j=0}^{k-1} L_j(\lambda\tau)\,. \end{aligned} \tag{15}$$

The final expression for the yield curve associated with the forward rate (13) is obtained by reordering the summation in equation (15) to get

$$\begin{aligned} y(\tau) \;\; &= \;\; \beta_1 + c_0\Big(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\Big) \\ &\qquad + e^{-\lambda\tau}\sum_{j=0}^{N-1}\Big(\sum_{k=j+1}^{N}\frac{c_k}{k}\Big)L_j(\lambda\tau)\,. \end{aligned} \tag{16}$$

The parameters β_1, λ and the coefficients $c_0, \cdots, c_N$ are now estimated from observations of bond prices by minimising the analogous form of expression (10) but modified to suit the specification (16). Of course, expression (16) is equivalent to the Nelson-Siegel form for the yield curve when $j = 0$ ($c_0 = \beta_2 + \beta_3$ and $c_1 = \beta_3$), but in general it provides a framework by which the Nelson-Siegel parametric form may be generalised to an arbitrary number of factors.

3.4 Summary

Observed bond prices at various maturations are used to construct an empirical yield curve. This yield curve is either modelled directly or is modelled indirectly by specifying a model for the forward rate of interest. These approaches are not equivalent. The long term interest rate is approached exponentially in the former but algebraically in the latter.

4 EMPIRICAL ILLUSTRATION

The models for the yield curve and forward rate proposed in Section 3 are now applied to gilts data published by the Bank of England. The data comprises weekly gilts yields for maturities ranging from 1 year (12 months) to 19 years (228 months) at intervals of 6 months, for the period from mid-January 1985 to the end of December 2004 (a total of 1042 weekly observations for each of the 37 maturities). These data are the smoothed yields constructed by the Bank of England from observed bond prices (see Anderson and Sleath, 1999) for a description of the procedures used to construct the Bank of England yield curve). Figure 1 shows 5 representative yield curves[1].

The first empirical question that needs to be addressed concerns the sensitivity of the Nelson-Siegel model to the choice of λ. Recall that Diebold and Li (2005) set $\lambda = 0.0609$ as this ensures that the medium-term factor $\big(1 - e^{-\lambda\tau}\big)/(\lambda\tau) - e^{-\lambda_1\tau}$ reaches a maximum value at a maturity of 30 months. Figure 2 illustrates the fit to the mean yield curve for three values of λ when applied to the Bank of England data. The fit for $\lambda = 0.01$ is acceptable, but the quality of this fit deteriorates rapidly as λ increases, and in particular, the fit is poor for $\lambda = 0.0609$.

Another way to illustrate the sensitivity of the fitted yield curve to the value of the parameter λ is to examine the squared residuals obtained by fitting equation (4) to the data. Figure 3 illustrates how the root mean squared difference between the observed mean yield curve and its model-based value depends on the value of λ for a 3-factor model of the yield curve based on equation (9) with $N = 1$ (dashed line), and a 3-factor model of the yield curve based on the forward rate in equation (13) with $N = 1$ (solid line).

It is apparent from Figure 3 that the optimal value of λ is very small and that the fit of the yield curve deteriorates sharply as λ increases. It is therefore reasonable to conjecture that treating λ as fixed in the estimation may be suboptimal. In effect, the model requires the estimation of λ, which in turn implies that the model is, in fact, a 4-factor model. Based on this evidence, the forecasting performance of a 3-factor

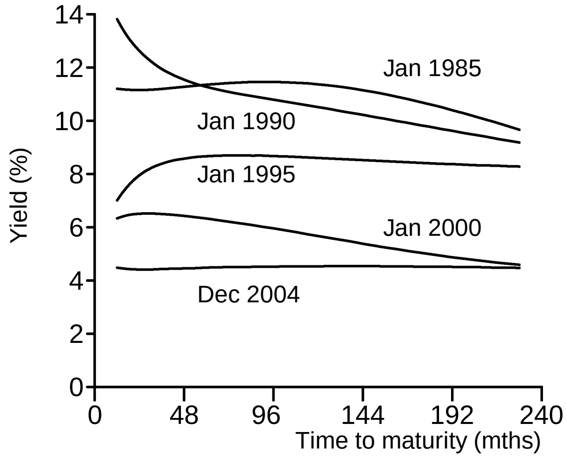

Figure 1. Yield curves constructed by the Bank of England from observed bond prices are shown for various dates for bonds with maturities from 1 year to 19 years.

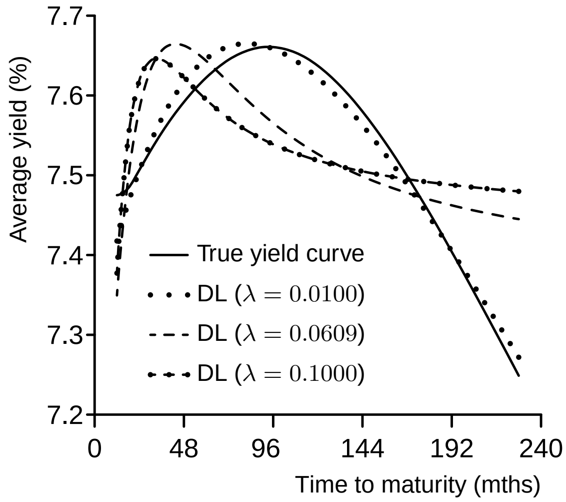

Figure 2. The mean yield curve (solid line) is shown together with the mean yield curves calculated from the model of Diebold and Li (DL) with $\lambda = 0.0100$ (dotted curve), DL's preferred choice of $\lambda = 0.0609$ (dashed line) and $\lambda = 0.1000$ (dot-dashed line).

[1]The data are for the weeks beginning 16 January 1985, 17 January 1990, 18th January 1995, 19 January 2000 and 29 December 2004.

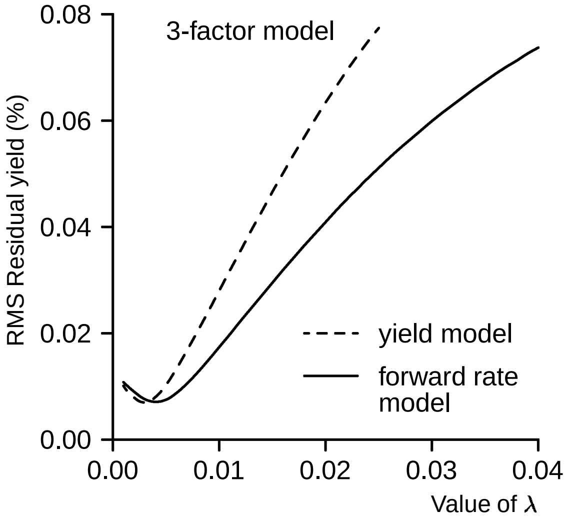

Figure 3. The root mean squared difference between the observed mean yield curve and its model-based value calculated from a 3-factor model of the yield curve (dashed line) and a 3-factor model of the forward rate (solid line) is shown for values of λ ranging from $\lambda = 0.001$ to $\lambda = 0.040$. Averages are calculated over 37 maturities from 12 months to 228 months at intervals of 6 months.

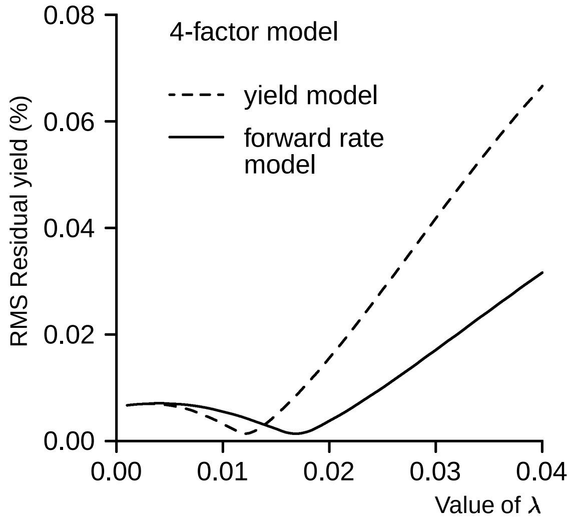

Figure 4. The root mean squared difference between the observed mean yield curve and its model-based value calculated from a 4-factor model of the yield curve (dashed line) and a 4-factor model of the forward rate (solid line) is shown for values of λ ranging from $\lambda = 0.001$ to $\lambda = 0.040$. Averages are calculated over 37 maturities from 12 months to 228 months at intervals of 6 months.

model will almost certainly be severely compromised.

The next question to be addressed is whether or not the introduction of additional linear factors can ameliorate the effect of fixing the value of λ. Figures 4 and 5 illustrate respectively how the root mean squared difference between the observed mean yield curve and its model-based value depends on the value of λ in the cases of a 4-factor ($N = 2$) model and a 5-factor ($N = 3$) model of the yield curve (dashed line) and forward rate (solid line).

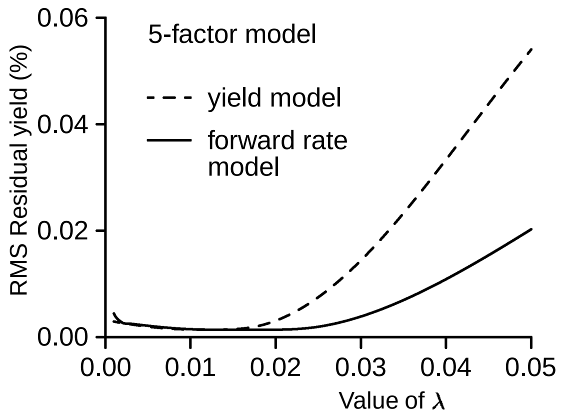

Figure 5. The root mean squared difference between the observed mean yield curve and its model-based value calculated from a 5-factor model of the yield curve (dashed line) and a 5-factor model of the forward rate (solid line) is shown for values of λ ranging from $\lambda = 0.001$ to $\lambda = 0.040$. Averages are calculated over 37 maturities from 12 months to 228 months at intervals of 6 months.

The first important conclusion to emerge from this work is that the extension of both models from a 3-factor model significantly increases the quality of the fit, reduces the sensitivity of the model to the value of λ and induces the optimal value of λ to increase. These affects are most marked for the move from 3 to 4 factors. It appears that the 5-factor model does not significantly improve the quality of the fit although the sensitivity of the fit to the value of λ is further reduced. Note that in all cases the curvature of the residual curve with respect to λ is always greater when the model is written for the yield curve as opposed to the forward rate. Therefore the quality of the fit to observed bond prices at various states of maturation is more sensitive to the value of the parameter λ when the model is written for the yield curve as opposed to the forward rate. Furthermore, the value of λ which minimises the residual function is always smaller when the model is written for the yield curve as opposed to the forward rate. In retrospect, one might argue that this result is predictable and is simply a consequence of the difference between a yield curve which decays exponentially to the long term interest rate as occurs when the model is written for the yield curve, and a yield curve which decays algebraically to the long term interest rate as occurs when the model is written for the forward rate. To accommodate the effect of long maturities, the former needs a smaller value of lambda than the latter to prevent premature decay of exponential terms and allow the model specifying the yield curve to better incorporate yield data from bonds with long maturities.

5 CONCLUSION

This article presents a generalisation of the Nelson-Siegel model for fitting yield curves. The method is based on the Fourier-Lageurre representation of a continuous function in $(0, \infty)$ and preserves the theoretical properties of the Nelson-Siegel approach. Once the time-scale parameter λ has been fixed, the model may be implemented in terms of simple least-squares framework.

The method is applied to UK gilts data and a number of robust conclusions emerge. The first of these is that the root mean squared difference between the observed mean yield curve and its model-based value appears to be a unimodal function of the parameter λ. This suggests that the optimal value of λ can be found by a simple search procedure. Once λ has been fixed, a 4-factor model is found to provide an acceptable fit to the data and the additional of further factors does not improve the quality of this fit. With respect to quality of the fit, there seems little to choose between modelling yields directly and modelling forward rates, but the latter is preferable because the quality of the fit is less sensitive to the choice of λ. It is conjectured that this result arises because the yield curve decays algebraically to the long-term interest rate when the model is written for the forward rate, whereas the decay is exponential when the model is written for yields.

The general approach to modelling the yield curve proposed in this article has interesting implications for future research. In terms of modelling the cross section of yields, this method should be compared to other smoothing procedures such as those based on splines. In terms of forecasting yields, this method lends itself to the dynamic factor approach proposed by Diebold and Li (2005).

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