Jump Diffusion Model: An Application to the Japanese Stock Market

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EXTENDED ABSTRACT

The Black-Scholes(BS) model has been widely and successfully used to model the return of asset and to price financial options. Despite of its success the basic assumptions of this model, that is, Brownian motion and normal distribution are not always supported by empirical studies. Those studies showed the two empirical phenomena: (1) the asymmetric leptokurtic features, (2) the volatility smile. The first means that the return distribution is skewed to the left and has a higher peak and two heavier tails than those of normal distribution, and the second means that if the BS model is correct, then the implied volatility should be flat. But the graph of the observed implied volatility curve often looks like the smile of the Cheshire cat. One of the causes for such phenomena is jumps in assets price processes. Figure 1 shows time series of 1 minute tick data of Yen/$ exchange rate between 15:00 and 24:00 on 21st of July 2005 when Chinese Yuan was revaluated. It seems obvious that there was a jump at the time of the revaluation. Many models were proposed to explain the two empirical phenomena. For example popular ones are normal jump diffusion model(Merton(1976)), stochastic volatility models(Heston(1993)), ARCH-GARCH models(Duan(1993)), etc. For other models see references in Kou(2002)). Among others we focus on a double exponential jump diffusion model proposed by Kou(2002) in this paper. Kou's model is very simple and has rich theoretical implication as described below: The logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. This model has the following advantages: (1) it can explain the two empirical phenomena, that is asymmetric leptokurtic feature and the volatility smile, (2) it leads to analytical solutions to many option-pricing problems. Despite of these advantages there are not many empirical studies based on this model partly because probability distribution function derived from this model is rather complicated and difficult to be estimated. However we fit Kou's model to Japanese stock data. Before doing so we applied Bipower test proposed by Barndorff-Nielsen and Shepard(2004) to see if Japanese stock price process contain jumps. After confirming jumps were existed we calculated option prices by the estimated Kou's and BS's model and compared those prices with the market price. As a result we found that Kou's model outperformed BS model.

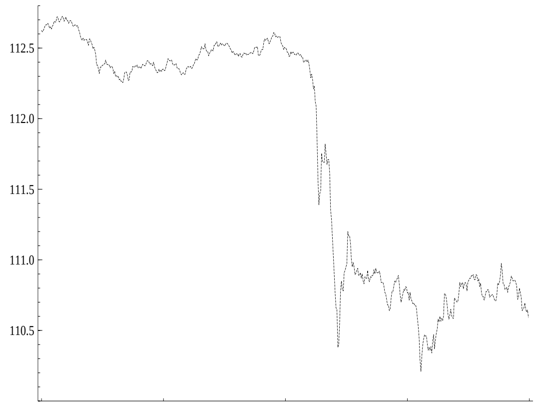


Figure 1. Jump effect of Chinese Yuan revaluation on Yen/$ exchange rate

The plan of this paper is as follows. In Section 2 we introduce the Barndorff-Nielsen and Shephard (2004), BN-S, hereafter, test which is a test to the adequacy of pure jump diffusion model (with no jumps) and we apply their test to real Japanese stock data in the subsection 2.2. In Section 3 we introduce Kou's model and its theoretical background in the subsection 3.1, and apply it to Japanese stock data to calculate option-prices in the subsection 3.2. In Section 4 we compare pure- and jump-diffusion models by observing volatility smile and other statistics and conclude this paper.

1. INTRODUCTION

It has been observed that structural changes and/or jumps often occur in financial time series data due to the policy changes and social events (see, for example, Wichern, Miller and Hsu (1976), Picard (1985), Inclán and Tiao (1994), Lee, Ha, Na and Na (2003), Lee and Na (2004), Lee, Tokutsu and Maekawa (2004)). In particular, if there is such a structural change, it is well known that a pure diffusion model does not provide a better fit to the financial data such as stock returns and interest rates. For this reason, jump diffusion models and Levy processes have recently applied to financial time series data. See Barndorff-Nielsen, Mikosch and Resnick (2001), Kou (2002), Shoutens (2003) and Cont and Tankov (2004). Recently BN-S proposed a new test for jump diffusion process based on bipower variation. In this section we apply their method to test whether an observed process is a jump diffusion process or a pure diffusion process(without jumps).

2. TESTING H_0: NO JUMP vs H_1: JUMP

2.1. Bipower Variation (BV) Test

Before we apply a jump diffusion model to real data we test the normality of return distribution for stock data contained in Nikkei 225 by using two tests, that is, Lilliefors test (abbreviated L-test hereafter) and Anderson-Darling test (abbreviated AD test hereafter). By conducting these two tests to 214 time series out of Nikkei 225 stock prices the normality assumption of the return distributions are not accepted by both of the tests. The results strongly suggest that those series may have jumps. Therefore we want to test if there are jumps in these series. As a test for the jump processes, we describe the method of Barndorff-Nielsen and Shephard (2004) test based on bipower variation. Their idea for constructing the test for jumps is based on the fact that the quadratic variation process of a semimartingale can be divided into the two parts: the continuous and discontinuous parts.

Let Y denote a semimartingale, and let Y^c and Y^d be the continuous part and the discontinuous part of Y, respectively. The definition of quadratic variation process of Y is given by

$$[Y]_t = p - \lim_{n\to\infty} \sum_{j=0}^{n-1} (Y_{t_{j+1}} - Y_{t_j})^2, \qquad (1)$$

for a sequence of partitions $t_0 = 0 < t_1 < \cdots < t_n = t$ such that $\sup_j\{t_{j+1} - t_j\} \to 0$ and $n \to \infty$. Then the following equation is known to hold

$$[Y]_t = [Y^c]_t + [Y^d]_t, \qquad (2)$$

where $[Y^d]_t = \sum_{0\le u\le t} \Delta Y_u^2$, where $\Delta Y_t = Y_t - Y_{t-}$ are jumps in Y. BN-S (2004) construct the test for jumps by checking whether $[Y]_t = [Y^c]_t$ holds or not. Namely, they consider the following hypothesis test:

H_0: Y is a member of the Brownian semimartingale ($\mathcal{BSM}$) class:

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \quad \text{vs.}$$

H_1: Y is a member of the $\mathcal{BSM}$ plus jump ($\mathcal{BSMJ}$) class:

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \sum_{j=1}^{N_t} c_j, \qquad (3)$$

where a_s and σ_s are càdlàg. W_t is a standard Wiener process, N_t is a counting process, which shows the number of jumps in the time interval $[0, t]$, and the jump size c_j is nonnegative random variables. It is worth mentioning, for later calculation, that the quadratic variation process of Y which belongs to $\mathcal{BSMJ}$ is given by

$$[Y]_t = \int_0^t \sigma_s^2 ds + \sum_{j=1}^{N_t} c_j^2 = [Y^c]_t + [Y^d]_t. \qquad (4)$$

To make the test feasible for the actual data, they introduce $Y_{\delta\lfloor t/\delta\rfloor}$ as the discretized version of Y, where $\delta > 0$ is a time interval, and $\lfloor t/\delta\rfloor$ is the largest integer that dose not exceed t/δ. For simplicity of the notation, let Y_δ stand for $Y_{\delta\lfloor t/\delta\rfloor}$. Let us consider Y_t to be the log-price of an asset, then the log return can be written as

$$y_j = Y_{j\delta} - Y_{(j-1)\delta}, \;\; j = 1, 2, \cdots, \lfloor t/\delta\rfloor. \qquad (5)$$

BN-S (2004) shows that the $1, 1$-order bipower variation process defined by

$$\{Y\}_t^{[1,1]} = p - \lim_{\delta\downarrow 0} \sum_{j=2}^{\lfloor t/\delta\rfloor} |y_{j-1}|\,|y_j|, \qquad (6)$$

can be expressed as

$$\{Y\}_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_s^2 ds = \mu_1^2 \, [Y^c]_t, \qquad (7)$$

if $Y \in \mathcal{BSMJ}$, where $\mu_1 = \mathrm{E}[|u|] = \sqrt{2/\pi} \simeq 0.79788$, $u \sim N(0,1)$. Therefore, we can see that $\mu_1^{-2}\{Y\}_t^{[1,1]} = [Y^c]_t$ and $[Y]_t - \mu_1^{-2}\{Y\}_t^{[1,1]} = [Y^d]_t$ (see (4) and (7)).
In calculating the quadratic variation $[Y]_t$ by

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2, \qquad (8)$$

and the bipower variation $\{Y\}_t^{[1,1]}$ by

$$\{Y_\delta\}_t^{[1,1]} = \sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}| \, |y_j|, \qquad (9)$$

$[Y_\delta]_t$ and $\{Y_\delta\}_t^{[1,1]}$ are said to be the realized quadratic variation and the realized bipower variation, respectively. Under the two assumptions: (a) The process σ^2 is pathwise bounded away from 0, and (b) The joint process (a, σ) is independent of the Wiener process W, they proposed the three kinds of test statistic for the above null hypothesis and calculated their asymptotic distributions as below.

$$\hat{G} = \frac{\delta^{-1/2}\left(\mu_1^{-2}\{Y_\delta\}_t^{[1,1]} - [Y_\delta]_t\right)}{\sqrt{\mu_1^{-4}\{Y_\delta\}_t^{[1,1,1,1]}}} \xrightarrow{L} N(0, \vartheta), \qquad (10)$$

$$\hat{H} = \frac{\delta^{-1/2}\left(\frac{\mu_1^{-2}\{Y_\delta\}_t^{[1,1]}}{[Y_\delta]_t} - 1\right)}{\sqrt{\{Y_\delta\}_t^{[1,1,1,1]} / \left\{\{Y_\delta\}_t^{[1,1]}\right\}^2}} \xrightarrow{L} N(0, \vartheta), \qquad (11)$$

$$\hat{J} = \frac{\delta^{-1/2}\left(\frac{\mu_1^{-2}\{Y_\delta\}_t^{[1,1]}}{[Y_\delta]_t} - 1\right)}{\sqrt{\max\left(t^{-1}, \{Y_\delta\}_t^{[1,1,1,1]} / \left\{\{Y_\delta\}_t^{[1,1]}\right\}^2\right)}} \xrightarrow{L} N(0, \vartheta), \qquad (12)$$

where ϑ is given by $\vartheta = \frac{\pi^2}{4} + \pi - 5 \simeq 0.6090$ and

$$\{Y_\delta\}_t^{[1,1,1,1]} = \frac{1}{\delta} \sum_{j=4}^{\lfloor t/\delta \rfloor} |y_{j-3}| \, |y_{j-2}| \, |y_{j-1}| \, |y_j| \xrightarrow{P} \mu_1^4 \int_0^t \sigma_s^4 ds, \qquad (13)$$

which is called the realized quadpower variation. If $Y \in \mathcal{BSMJ}$, it is shown that $\mu_1^{-2}\{Y_\delta\}_t^{[1,1]} - [Y_\delta]_t \leq 0$ and $\frac{\mu_1^{-2}\{Y\}_t^{[1,1]}}{[Y_\delta]_t} - 1 \leq 0$ (For details see Barndorff-Nielsen and Shephard (2004)). Hence, we reject the null hypothesis if the test statistic is significantly negative.

As asymptotic normality of Bipower variation test is valid for very large sample size such as a sample size in high frequency data we have to confirm if it is applicable to a medium sample size data such as daily data. To do so we performed the following simulation. We generated 1000 jump diffusion processes with 250 observations by Kou model and applied BV test to test null of no-jump. As a result we correctly rejected the null more than 950 times out of 1000 iterations. From this simulation experiment we may say that BV test is applicable to our daily data of Japanese stock data.

2.2. Empirical Study Using Japanese Stock Data

In this subsection, we perform the BN-S test for the Nikkei 225 index. Consequently, we obtain the result that the null hypothesis is rejected at the 5% significant level. Therefore, it is appropriate to consider a jump diffusion process for the Nikkei 225 index. This can be confirmed from the return distribution. In general, it is known that the distribution of empirical returns has two characteristics; fat tail (or excess kurtosis) and asymmetry. In particular, we can mention the existence of jumps in price processes as one of reasons why fat tail distribution can be observed. Here we employ a jump diffusion model proposed by Kou(2002) because it can capture two such characteristics and provide analytical formulas for prices of options. The model consists of two parts: (1) a geometric Brownian motion.(2) a compound Poisson process with jump sizes following a double exponential distribution. Using the approximate density of returns given by Kou's model, we show the goodness of fit of the density to actual returns along with the normal density in Figure 2. We can see that the density given by Kou's model shows better fit than the normal around the center and in tails(see Figure 3).

3. KOU MODEL

In this section we describe Kou's jump diffusion model.

3.1. Model Specification

Under probability measure $\mathbf{P}$ we assume that underlying asset price process $S(t)$ follows

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right), \tag{14}$$

where $W(t)$ is standard Brownian motion, $N(t)$ is a Poisson process with intensity λ and $\{V_i\}\mathcal{Y}$ is a i.i.d. nonnegative stochastic sequence. Again $\Upsilon = \log(V)$ is an asymmetric double exponential distribution with density

$$f_\Upsilon(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y\geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \quad \eta_1 > 1, \eta_2 > 0,$$

where $p, q \geq 0,\ p + q = 1$ are up-move jump and down-move jump respectively. Put another way,

$$\log(V) = \Upsilon \stackrel{d}{=} \begin{cases} \xi^+, & \text{with probability } p \\ -\xi^-, & \text{with probability } q \end{cases} \tag{15}$$

where ξ^+ and ξ^- is exponential random variable with mean $1/\eta_1$ and $1/\eta_2$. Note that $\stackrel{d}{=}$ denotes identically distributed. In this model we assume that stochastic element $N(t)$, $W(t)$, Υ_S are independent. For notational convenience and explicit solution for option price we assume that drift term μ and diffusion term σ are constants and restrict ourselves to one dimensional case. However these assumptions are easily generalized to more complex case.

Given a solution of SDE(14), then we obtain asset price dynamics

$$\begin{aligned} S(t) =& S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t \right. \\ & \left. + \sigma W(t)\right\} \prod_{i=1}^{N(t)} V_i, \end{aligned} \tag{16}$$

where $\mathrm{E}(\Upsilon) = \frac{p}{\eta_1} - \frac{q}{\eta_2}$, $\mathrm{Var}(\Upsilon) = pq(\frac{1}{\eta_1} + \frac{1}{\eta_2})^2 + (\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2})$ and

$$\begin{aligned} \mathrm{E}(V) &= \mathrm{E}(e^\Upsilon) \\ &= q\frac{\eta_2}{\eta_2 + 1} + p\frac{\eta_1}{\eta_1 - 1}, \quad \eta_1 > 1, \quad \eta_2 > 0. \end{aligned}$$

Again $\eta_1 > 1$ guarantees $\mathrm{E}(V) < \infty$ and $\mathrm{E}(S(t)) < \infty$. This means that average rate of up-jump does not exceed 100%.

Rate of return on Δt is given by (16) and

$$\begin{aligned} \frac{\Delta S(t)}{S(t)} =& \frac{S(t+\Delta t)}{S(t)} - 1 \\ =& \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t \right. \\ & + \sigma(W(t+\Delta t) - W(t)) \\ & \left. + \sum_{i=N(t)+1}^{N(t+\Delta t)} \Upsilon_i \right\} - 1. \end{aligned}$$

If Δt is sufficiently small, by omitting higher order term than Δt and using expansion $e^x \approx 1 + x + x^2/2$, one can approximate rate of return to the distribution

$$\frac{\Delta S(t)}{S(t)} \approx \mu\Delta t + \sigma Z\sqrt{\Delta t} + B \cdot \Upsilon$$

where Z and B are random variable of standard normal and binomial respectively, and $\mathbf{P}(B = 1) = \lambda\Delta t$, $\mathbf{P}(B = 0) = 1 - \lambda\Delta t$ and Υ is given by (15). The density function is

$$\begin{aligned} g(x) =& \frac{1-\lambda\Delta t}{\sigma\sqrt{\Delta t}}\phi\left(\frac{x - \mu\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ & + \lambda\Delta t\left\{p\eta_1 e^{(\sigma^2\eta_1^2\Delta t)/2} e^{-(x-\mu\Delta t)\eta_1} \right. \\ & \times \Phi\left(\frac{x - \mu\Delta t - \sigma^2\eta_1\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ & + q\eta_2 e^{(\sigma^2\eta_2^2\Delta t)/2} e^{-(x-\mu\Delta t)\eta_2} \\ & \left. \times\ \Phi\left(-\frac{x - \mu\Delta t + \sigma^2\eta_2\Delta t}{\sigma\sqrt{\Delta t}}\right)\right\} \end{aligned} \tag{17}$$

where $\phi(\cdot)$ is density function of standard normal and $\Phi(\cdot)$ is its distribution function.

3.2. Option Pricing by Kou Model

In this subsection we demonstrate Kou's formula of option pricing for European call given in Theorem 1 below. For obtaining option price we need to consider the sum of normal and double exponential distributions. Fortunately we can compute explicitly the distribution by using Hh function, which is a special function of mathematical physics, for more detail see Abramowitz and Stegun (1972, p. 691).

For a probability $\mathbf{P}$ we define

$$\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) := \mathbf{P}\{Z(T) \geq a\},$$

where $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} \Upsilon_i$, Υ follows double exponential distribution with density

$f_\Upsilon(y) \sim p \cdot \eta_1 e^{-\eta_1 y}\mathbf{1}_{\{y\geq 0\}} + q \cdot \eta_2 e^{y\eta_2}\mathbf{1}_{\{y<0\}}$ and $N(t)$ is a Poisson process with intensity λ. This Υ is the formula for European call option, which given by the sum of Hh function. As for the explicit form of Υ, it is given by the following theorem:

Theorem 1 (Kou (2002)) *The European call price is given by*

$$\begin{aligned}\psi_c(0) =& S(0)\Upsilon(r + \frac{1}{2}\sigma^2 - \lambda\zeta,\\ &\sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta_1}, \tilde{\eta_2}; \log(K/S(0)), T)\\ &- Ke^{-rT}\Upsilon(r - \frac{1}{2}\sigma^2 - \lambda\zeta,\\ &\sigma, \lambda, p, \eta_1, \eta_2; \log(K/S(0)), T), \quad (18)\end{aligned}$$

where

$$\tilde{p} = \frac{p}{1+\zeta} \cdot \frac{\eta_1}{\eta_1 - 1}, \quad \tilde{\eta_1} = \eta_1 - 1, \quad \tilde{\eta_2} = \eta_2 + 1,$$
$$\tilde{\lambda} = \lambda(\zeta + 1), \quad \zeta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

Note that when substituting Φ for Υ the equation (18) seems like Black-Scholes formula for European call. For the proof of Theorem 1, see Theorem 3 in Kou and Wang (2004).

4. COMPARISON OF OPTION PRICES: BS vs. KOU

In this section we compare option prices derived from BS formula and Kou's formula (18) as well as implied volatility derived from BS model and Kou's model by using 214 series with more than 1000 observations out of Nikkei 225 from 1 June 1992 to 31 December 2002. We estimated parameters of Kou's density function (17) by MLE and substituted the estimators to (18) to obtain the European call option prices for each stock. We also calculated the option price for each stock by BS formula. Finally we compared the three prices: the market price (MP), theoretical prices derived by BS and Kou's models. We used the market prices of European call option for Nikkei 225 from September 10, 1999 to December 12, 2002 with various strike prices and times to maturity. The relative differences of theoretical and market prices divided by market prices are shown in the figures 4-5. In each figure the vertical axis denotes the difference between the two prices and horizontal axes denote strike price and time to maturity. Figures 4 and 5 show the differences between the market price and theoretical price by Kou's mode and the differences between the market price and theoretical price by BS model. These figures show that the calculated prices by Kou's model are much closer to the real data than calculated prices by BS model. To see this we calculated a measure of distance, i.e., average relative percentage error(ARPE) defined by

$$\text{ARPE} = \frac{1}{M}\sum_{i=1}^{M}\left|\frac{\hat{C}_i - C_i}{C_i}\right|,$$

where M is the number of options, and C_i and $\hat{C}_i$ denote market price and model price, respectively. Table 1 shows the results.

Table 1. Distances from the market price

	BS	Kou	sample size
ARPE	0.3646	0.2895	50955

5. CONCLUSION

In this paper we apply bipower test proposed by Barndorff-Nielsen and Shephard (2004) to Japanese stock data and the test shows that null of no jump is often rejected in Japanese stock price series. Many jump diffusion models are studied in the literature. Among others we choose Kou's jump diffusion model by reasons mentioned in Kou (2002) and we fit the model to Japanese data. To see the performance of this model we compare option prices by Kou's and Black-Scholes model with the market prices. From our data analysis we conclude that Kou's model is fitted better than Black-Scholes model to Japanese market.

6. REFERENCES

Abramowitz, M., and I. Stegun. 1972. *Handbook of Mathematical Functions.* Dover, New York.

Barndorff-Nielsen, O.E., Mikosch, T. and Resnick, S. I. 2001. *Lévy Processes. Theory and Applications.* Birkhäuser, Boston.

Barndorff-Nielsen, O.E. and Shephard, N. 2004. Econometrics of testing for jumps in financial economics using bipower variation. Unpublished discussion paper: Nuffield College, Oxford.

Black, F. and M. S. Scholes. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81: 637-654.

Cont, P. and Tankov, P. 2004. *Financial Modelling with Jump Processes.* Chapman & Hall/CRC, London.

Duan, J.-C. 1995. The GARCH Option Pricing Model. *Mathematical Finance* 5 (1): 13-32.

Heston, Steven L. 1993. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The review of Financial Studies* 6 (2): 327-343.

Inclán, C. and Tiao, G. C. 1994. Use of cumulative sums of squares for retrospective detection of changes of variances. *Journal of the American Statiscal Association* 89: 913-923.

Kou, S. G. 2002. A jump diffusion model for option pricing. *Management Science* 48: 1086-1101.

Kou, S. G. and H. Wang. 2004. Option pricing under a double exponential jump diffusion model. *Management Science* 50: 1178-1192.

Lee, S., Ha, J., Na, O. and Na, S. 2003. The cusum test for parameter change in time series models. *Scand. J. Statist* 30: 781-796.

Lee, S. and Na, O. 2004. Test for parameter change in stochastic processes based on conditional least squares estimator. *To appear in J. Multi. Anal.*

Lee, S., Tokutsu, Y. and Maekawa, K. 2004. The residual cusum test for parameter change in regression models with ARCH errors. *To appear in J. Japan Statist. Soc.*

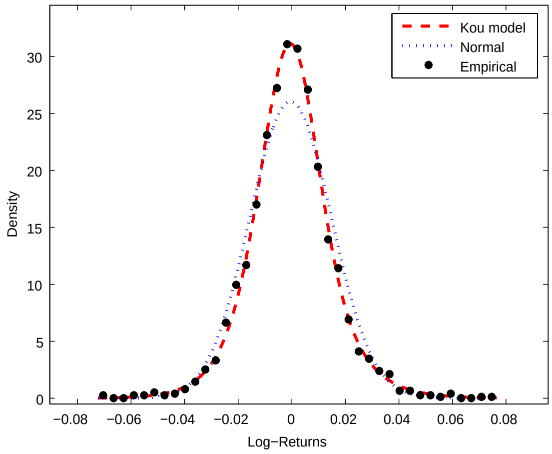
Lee, S., Wee, K. 2004. Residual Empirical Process for Diffusion Processes. *Submitted for publication.*

Merton, R. C. 1976. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3: 125-144.

Picard, D. 1985. Testing and estimating change-points in time series. *Advances in Applied Probability* 17: 841-867.

Schoutens, W. 2003. *Lévy Processes in Finance.* Wiley, New York.

Wichern, D.W., Miller, R.B. and Hsu, D.A. 1976. Changes of variance in first-order autoregressive time series models - with an application. *Applied Statistics* 25: 248-256.



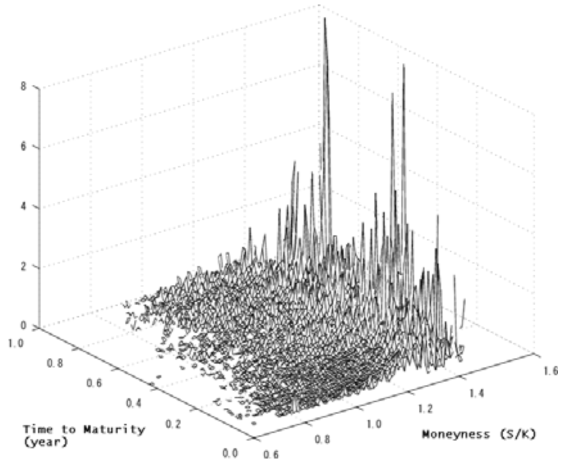
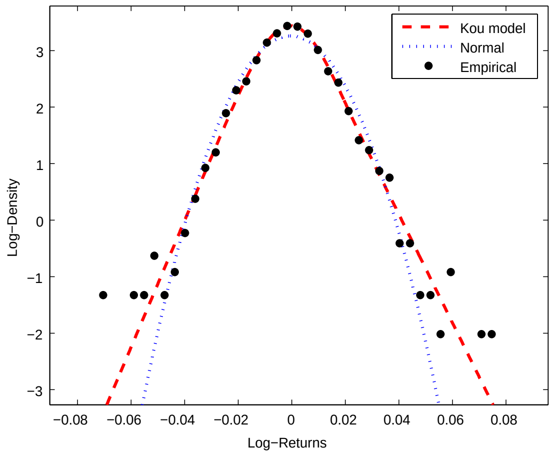


Figure 2. The densities of Empirical, Kou model and Normal for the Nikkei 225 index

Figure 4. Relative Difference between MP and BS Price



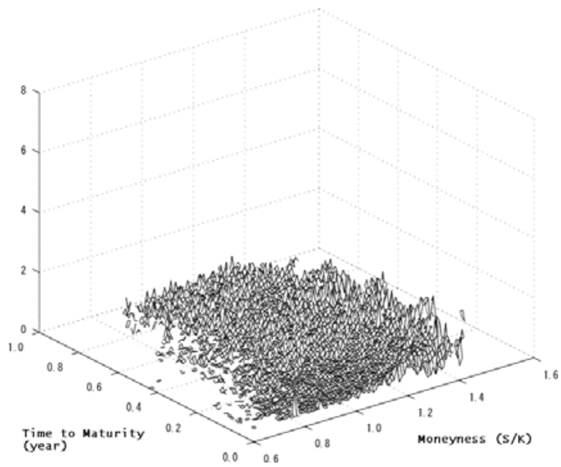


Figure 3. The log-densities of Empirical, Kou model and Normal for the Nikkei 225 index

Figure 5. Relative Difference between MP and Kou Price