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Proximity termination conditions for two aircraft: one with circular and one with straight uniform motion

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Abstract. In practical operations aircraft may be considered to be either in turning or in straight flight. Application of the Frenet-Serret theorem to the aircraft trajectories (flight-paths) also demonstrates this characteristic. The Closest Point of Approach (CPA) between two trajectories is the point at which the relative displacement (range) between the two aircraft is at a minimum. There has been long standing use of this measure for resolving situations of close proximity both for aircraft and for ships (Britt, Schrader, 1970). Also well identified in international literature is the use of this measure as the basis of several guidance laws (Gazit, Powell, 1996; Zeghal, 1998). The determination of the location of the CPA therefore emerges as an important design calculation for the implementation of many proximity management functions (Fulton, 2002; Krozel and Peters, 1997; Merz, 1973a; Merz, 1973b; Miele et al, 1999; Tarnopolskaya and Fulton, 2009).

This paper is a sequel to Fulton and Huynh (2009) that also contributes towards the development of a generalised CPA concept. There, the determination of the CPA for two aircraft each with linear motion was presented and the solution methodology was discussed. In this paper the situation is presented where own-aircraft has a circular motion and an intruder aircraft has a linear motion. The generalised geometrical technique developed in Huynh and Fulton (2007) is extended and applied to encounters comprised of 3D trajectories. Further geometrical characterizations of Fermat's method for stationary points in vector form that are specific to this problem are provided. Again the formulation leads to the identification of a fixed reference point for the stationary states. This point, determined by three conditions, is located on a straight line that:

1. lies within the plane of the first aircraft turn circle,
2. passes through the centre of that turning circle, and
3. is orthogonal to the flight path of the second aircraft.

This reference point can then be used to determine the location of the CPA. The analysis provides a very effective method for determining the CPA that can be easily applied in an operational context. For example, the situation modelled can occur during normal operations where one aircraft is on a straight-in approach to a runway and another aircraft is turning onto final approach for a second parallel runway. In this case the model that projects a linear extrapolation of each aircraft's velocity vector will not identify where in the turn the true CPA will occur. The present methodology can determine the actual CPA and this methodology can then be incorporated into an aircraft's flight management system to warn pilots as to where in the varying approaches vigilance needs to be the greatest.

As a consequence of this work more accurate and more concise specifications of aircraft proximity management functions can be achieved. These specifications are then useful when developing robust and dependable algorithms for aircraft avionics. The methodology is also considered to be useful in other forms of vehicle navigation such as robotics.

Keywords: *Collision avoidance, cooperative manoeuvres, fixed reference point, optimisation, turn rates, proximate.*

1. FORMULATION AS AN OPTIMISATION PROBLEM

The problem of two aircraft moving, one with a circular uniform motion and one with a straight uniform motion, can be posed in the earth reference frame as follows:

Assume that aircraft A is turning around a circle $\mathbb{C}_A$ with centre O_A and radius a. Aircraft B moves on a straight line with the flight path Δ. The XY-plane is generally defined as the plane of the turning circle of aircraft A. Let line $\Delta^{'}$ be the image of the line Δ by the normal projection of Δ onto the XY-plane. Let d be the minimal distance between the turn centre O_A and the line $\Delta^{'}$. The coordinate system's origin is set at the turn centre O_A. The X-axis $\left(\overrightarrow{O_A X}\right)$ is determined by a straight line in the XY-plane which is drawn from the turn centre O_A and orthogonal to the line Δ. Thus, the X-axis is perpendicular to $\Delta^{'}$ and intersects with $\Delta^{'}$ at point O_B. Consequently, the Y-axis $\left(\overrightarrow{O_A Y}\right)$ is determined by the perpendicular line to the X-axis at the turn centre O_A. Clearly, the Y-axis is parallel to $\Delta^{'}$. Finally, the Z-axis $\left(\overrightarrow{O_A Z}\right)$ is perpendicular to the XY-plane and determined by the Right-Hand rule, as shown in Figure 1.

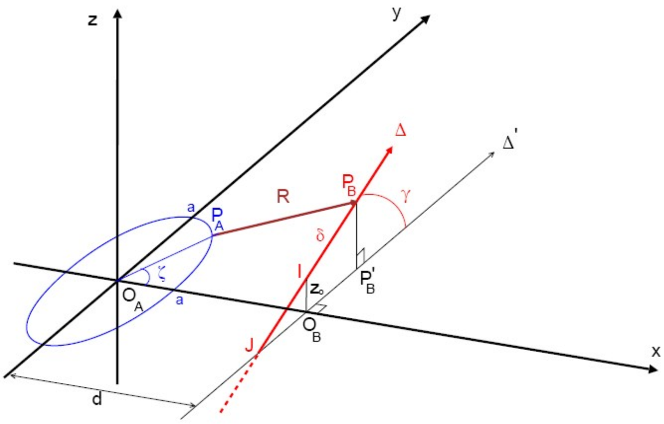

Figure 1: Cartesian Coordinates System

By the well-known mathematical convention, the left turn (anti-clockwise) is in the positive direction for angles. Let ζ be the turning angle of aircraft A, measured anti-clockwise from the X-axis $\left(\overrightarrow{O_A X}\right)$. Assume that aircraft A is uniformly turning with a constant angular velocity ω_A, thus $\zeta = \omega_A t + \zeta_0$ where ζ_0 is a constant. Let $P_A = (x_A, y_A, z_A)$ be an arbitrary location of aircraft A on its turn circle.

Clearly, $P_A = (a\cos\zeta, a\sin\zeta, 0)$.

Now, let $J \in \Delta \cap \Delta^{'}$ and I be the point on Δ such that O_B is the foot of I on the XY-plane by the Z-projection ($I \in \Delta$ with z_0 is the Z-coordinate of I). Clearly, $O_B = (d, 0, 0)$, $I = (d, 0, z_0)$ and

$$J = \begin{cases} O_B & \text{for } \Delta \perp C_A \\ (d, z_0 \tan\gamma, 0) & \text{otherwise} \end{cases}$$ (where γ is the angle of the line Δ and the XY-plane)

If $P_B \in \Delta$ then $P_B = (x_B, y_B, z_B)$ where $x_B = d$, $y_B = \delta\cos\gamma$, $z_B = \delta\sin\gamma + z_0$ and $\delta = V_B t + \delta_0$ where V_B and δ_0 are constants for the straight and uniform motion.

Furthermore, as indicated in Figure 1, we have: $O_A = (0,0,0)$, $O_B = (d,0,0)$ and $P_B^{'} = (d, y_B, 0)$.

Let $\underline{d} = \overrightarrow{O_A O_B}$ with $d = \|\underline{d}\|$, $\underline{z_0} = \overrightarrow{O_B I}$ with $|z_0| = \|\underline{z_0}\|$, $\underline{\delta} = \overrightarrow{IP_B}$ with $|\delta| = \|\underline{\delta}\|$, $\underline{a} = \overrightarrow{O_A P_A}$ with $a = \|\underline{a}\|$ and $\underline{R} = \overrightarrow{P_A P_B}$ with $R = \|\underline{R}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$.

Suppose we are given the vectors: $\underline{a}$, $\underline{d}$, $\underline{\delta}$ and $\underline{z_0}$. Then, the relative range ($\underline{R}$) can be presented by:

$$\underline{R} = -\underline{a} + \underline{d} + \underline{z_0} + \underline{\delta}$$

Thus, $$R^2 = a^2 + d^2 + z_0^2 + \delta^2 - 2ad\cos\zeta - 2a\delta\cos\gamma\sin\zeta + 2\delta z_0\sin\gamma$$

$$\text{where } \begin{cases} \zeta = \omega_A t + \zeta_0 \\ \delta = V_B t + \delta_0 \end{cases}$$

Here, if the initial values (ζ_0, δ_0) are given, then the miss-distance of two aircraft (one with a circular motion and one with a straight motion) is defined as the minimum value of R . In this case, a minimisation for R is equivalent to a minimisation for R^2 (for $R \geq 0$). Thus, the miss-distance of aircraft can be determined if we can find values of ζ and δ so as to minimise R^2 . Clearly, this is a classical nonlinear and unconstrained optimisation problem. In particular, the total differential (Caunt, 1946) of the function R^2 is used to find the geometrical relation of stationary points and to understand the nature of the problem.

2. DETERMINATION OF AIRCRAFT LOCATIONS FOR MISS-DISTANCES

In this section, the geometrical relationship between the aircraft at the CPA is derived and discussed. Some basic cases of the possible proximity termination conditions are presented.

2.1. Fermat's equation and the fixed reference point for stationary states

From the previous section, the squared distance between two aircraft is a function of two variables ζ and δ . Hence, the total differential of R^2 , with respect to ζ and δ , is given by:

$$dR^2 = \frac{\partial R^2}{\partial \zeta} d\zeta + \frac{\partial R^2}{\partial \delta} d\delta \qquad \text{with} \begin{cases} \zeta = \omega_A t + \zeta_0 \\ \delta = V_B t + \delta_0 \end{cases}$$

$$\text{where} \quad \frac{\partial R^2}{\partial \zeta} = 2ad\sin\zeta - 2a\delta\cos\gamma\cos\zeta$$

$$\frac{\partial R^2}{\partial \delta} = 2\delta - 2a\cos\gamma\sin\zeta + 2z_0\sin\gamma$$

Thus, the total derivative (Matlak, 1961) of R^2 can be described as:

$$\frac{dR^2}{dt} = \frac{\partial R^2}{\partial \zeta}\frac{d\zeta}{dt} + \frac{\partial R^2}{\partial \delta}\frac{d\delta}{dt} \quad \Rightarrow \quad \frac{dR^2}{dt} = \frac{\partial R^2}{\partial \zeta}\omega_A + \frac{\partial R^2}{\partial \delta}V_B$$

Fermat's method for stationary points (Sanford, 1930; Ball, 1960; Paolini, 2003) states that stationary points of R^2 can be found as the roots of the following equation:

$$\frac{dR^2}{dt} = 0 \quad \Rightarrow \quad \frac{\partial R^2}{\partial \zeta}\omega_A + \frac{\partial R^2}{\partial \delta}V_B = 0$$

This is Fermat's equation for stationary points for the problem at hand (and in short, in this paper, Fermat's equation). In terms of ζ and δ variables, the above equation can be written as:

$$\omega_A\left(ad\sin\zeta - a\delta\cos\gamma\cos\zeta\right) + V_B\left(\delta - a\cos\gamma\sin\zeta + z_0\sin\gamma\right) = 0$$

Now, let $W = \dfrac{\omega_A}{V_B}$, Fermat's equation becomes:

$$W\left(ad\sin\zeta - a\delta\cos\gamma\cos\zeta\right) + \delta - a\cos\gamma\sin\zeta + z_0\sin\gamma = 0$$

$$\Rightarrow \qquad W\left(y_A d - x_A y_B\right) + \frac{y_B}{\cos\gamma} - y_A\cos\gamma + z_0\sin\gamma = 0$$

$$\Rightarrow \qquad y_A\left(d - \frac{\cos\gamma}{W}\right) + \frac{y_B}{W\cos\gamma} - x_A y_B = -\frac{z_0\sin\gamma}{W}$$

Thus, Fermat's equation can be cast in the form:

$$\begin{vmatrix} x_A & y_A & 1 \\ \dfrac{1}{W\cos\gamma} & 0 & 1 \\ d + \dfrac{\sin^2\gamma}{W\cos\gamma} & y_B & 1 \end{vmatrix} = -\frac{z_0\sin\gamma}{W}$$

This is the determinantal form of Fermat's equation for the general case of this problem.

Let $\quad k = \dfrac{\sin^2\gamma}{W\cos\gamma}$, $H = -\dfrac{z_0\sin\gamma}{W}$, $F = \left(\dfrac{1}{W\cos\gamma}, 0, 0\right)$, $P_L = \left(d + k, y_B, 0\right)$ and

$$L = \left\{\left(x, y, z\right):\ x = d + k,\ \mathrm{y} \in \mathbb{R},\ z = 0\right\}$$

Therefore, the vector form of Fermat's equation is given by: $\overrightarrow{FP_A} \times \overrightarrow{FP_L} = H\hat{z}$ where $\hat{z}$ is the unit vector of Z-axis. Note that the point F is a fixed reference point for stationary states.

Finally, this result gives rise to some exceptional cases because, in this form, Fermat's vector equation is undefined for the cases where: $\cos\gamma = 0$ and $W = 0$ (that is, $\omega_A = 0$). These cases can be considered separately by recasting the equation and by then applying a similar approach.

2.2. Geometrical solution for the general case

Consider two possible situations:

(i) P_A is given, and

(ii) P_B is given.

2.2.1. P_A is given

If P_A is given then $\overrightarrow{FP_A}$ is determined. Define the line L' as a parallel line to $\overrightarrow{FP_A}$ and separated from $\overrightarrow{FP_A}$ by a distance $h = \dfrac{|H|}{\left\|\overrightarrow{FP_A}\right\|}$. There are two possible cases for the line L' produced by this definition.

However, only one $L^{'}$ may be selected to satisfy the vector equation: $\overrightarrow{FP_A} \times \overrightarrow{FP_L} = H\hat{z}$, where $P_L \in L \cap L^{'}$ and $\hat{z}$ is the unit vector of Z-axis. Hence, the point $P_B^{'}$ is also determined, with $P_B^{'} \in \Delta^{'}$ and the Y-coordinate of $P_B^{'}$ is equal to the Y-coordinate of P_L.

Finally, because of $P_B^{'}$ is the Z-projection of P_B on the XY-plane, therefore P_B is the minimal solution of R. The geometrical interpretation is shown in Figure 2.

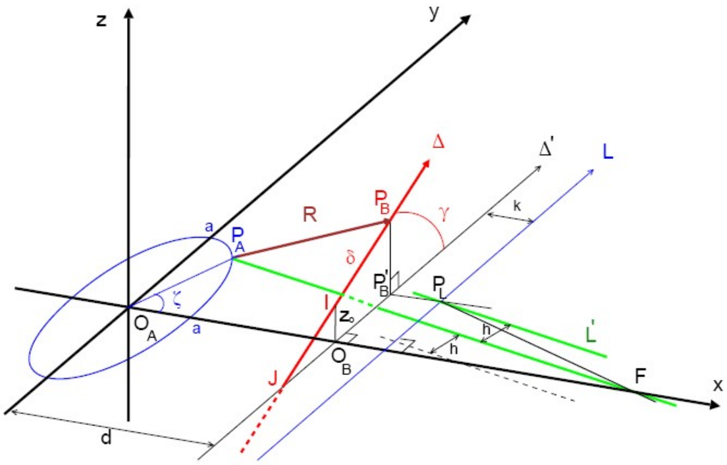

Figure 2: Geometrical solution for the general case when P_A is given

2.2.2. P_B is given

If P_B is given then $P_B^{'}$ is known. Thus, the point P_L is also determined. Now, let $h = \frac{|H|}{\left\|\overrightarrow{FP_L}\right\|}$. Next, draw a circle $\mathbb{C}_L$ with centre P_L and radius h. Then, from point F draw two tangents to the circle $\mathbb{C}_L$. And then, select the tangent T such that: $\overrightarrow{FP_A} \times \overrightarrow{FP_L} = H\hat{z}$, where $P_A \in T \cap \mathbb{C}_A$ and $\hat{z}$ is the unit vector of Z-axis. Finally, P_A is determined to minimise the relative distance R if $T \cap \mathbb{C}_A \neq \varnothing$.

The geometrical solution is presented in Figure 3.

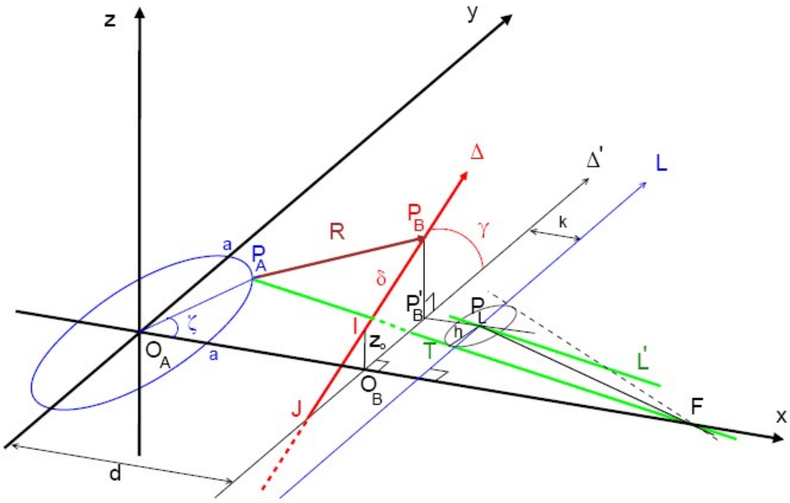

Figure 3: Geometrical solution for the general case when P_B is given

2.3. Geometrical solution for the planar case ($\sin\gamma = 0$)

This is a simple case commonly found in the aircraft or ships motion. If $\sin\gamma = 0$ then the flight path Δ is parallel to the plane of the turn circle $\mathbb{C}_A$. Clearly, the Z-coordinate of point $P_B \in \Delta$ is constant ($z_B = z_0$); therefore the solution for this 3D optimisation problem is equivalent to its planar case (in 2D) produced by Z-projection. Thus, the determinantal form of Fermat's equation is reduced to:

$$\begin{vmatrix} x_A & y_A & 1 \\ \frac{1}{W} & 0 & 1 \\ d & y_B & 1 \end{vmatrix} = 0$$

Let $F = \left(\frac{1}{W}, 0, 0\right)$ and $P_L = (d, y_B, 0) = P_B^{'} = P_B$ (P_L, P_B and $P_B^{'}$ are unique).

Then Fermat's vector equation is presented by: $\overrightarrow{FP_A} \times \overrightarrow{FP_B} = 0$

Clearly, three points: P_A, F and P_B are collinear. The fixed reference point F (for stationary states) is independent to the locations of aircraft: P_A and P_B. Thus, if one of P_A or P_B is given and then the one that remains can be determined so as to minimise the relative distance R.

Note that if point F lies outside the aircraft A's turn circle $\mathbb{C}_A$ then the sufficient condition to construct point P_A (the minimal solution for R) is: the given point $P_B \in \Delta$ must be bounded by two tangent lines which are drawn from the fixed reference point F to the circle $\mathbb{C}_A$.

Finally, the geometrical solution is shown in Figure 4.

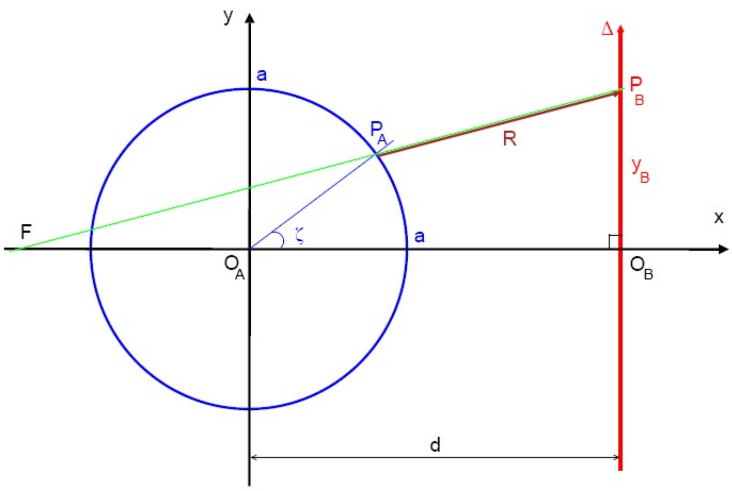

Figure 4: Geometrical solution for the planar case ($\sin\gamma = 0$)

3. DISCUSSION

In this paper, a geometrical method to find the CPA between two aircraft where one is moving with a circular uniform motion and one with a straight uniform motion is discussed and demonstrated. Solutions for the general case and the planar case are presented in detail. The exceptional cases have been identified. The geometrical solutions for these can be directly derived from Fermat's equation as shown in the following determinantal forms (by inspection):

(i) the case of $\cos\gamma = 0$ (orthogonal flight paths): $\begin{vmatrix} 0 & y_A & 1 \\ \dfrac{d}{Wd+1} & 0 & 1 \\ d & \delta & 1 \end{vmatrix} = 0$, and

(ii) the case of $\omega_A = 0$: $\begin{vmatrix} 0 & y_A & 1 \\ d & 0 & 1 \\ d\sin^2\gamma & y_B & 1 \end{vmatrix} = -z_0 \sin\gamma \cos\gamma$

Finally, the approach demonstrated has also been used to find the CPA for the case of where both aircraft are turning in a generally oriented 3D circle.

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