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Some new approximation results for utilities in revealed preference theory

Eberhard, A.C.[1], **S. Schreider**[1], **L. Kocoska (ne Stojkov)**[1], **J–P. Crouzeix**[2] **and D. Ralph**[3]

[1] *School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, VIC., Australia.*
[2] *LIMOS, Universite Blaise Pascal, de Clermont, France*
[3] *Judge Business School, University of Cambridge, UK.*
Email: andy.eb@rmit.edu.au

Abstract: A *utility* function $u : \mathbf{R}^n_+ := \left\{x \in \mathbf{R}^n \mid x^i \geq 0 \text{ for all } i\right\} \to \underline{\mathbf{R}} := \mathbf{R} \cup \{-\infty\}$ is often used to reflect the preference structure with respect to possible consumption of n commodities, denoted by a vector $x \in \mathbf{R}^n_+$ each of which are priced according to an associated vector of prices $p \in \mathbf{R}^n_+$. We assume that it may take the value $-\infty$ so as to allow for implicit constraints in the framing of associated optimization problems. It is well known that one can define a preference relation $y\mathcal{R}x$, "y is preferred to x", via a utility using $y \in S_{-u}(x) := \left\{z \in \mathbf{R}^n_+ \mid -u(z) \leq -u(x)\right\}$. It is natural to assume u is non-decreasing and so $u(x_1) \leq u(x_2)$ when $x_1 \leq x_2$. Clearly for any strictly increasing monotone function $k : \underline{\mathbf{R}} \to \underline{\mathbf{R}}$ we have $S_{-k \circ u} = S_{-u}$, allowing one to rescale the utility and leading an inherent lack of uniqueness.

When dealing with consumer demand in economic modeling, researchers often solve the optimization problem which maximises the utility for a given budget constraint. The real data on consumption are used to fit parameters of various a-priori prescribed mathematical functions which demonstrate constant return to scale and thus pose as utilities. An alternative approach to the fitting of a utility function (Eberhard et al. (2007)) allows the raw data to determine the functional form of the utility. This approach is motivated by the desire to allow the data to influence the corresponding price and quantity-price elasticities that are needed to be estimated using real data on consumption preference (Kocoska et al. (2009)) and (Eberhard et al. (2009c)).

In econometrics linear regression is used for the computing of elasticities but the approach used in (Kocoska et al. (2009)) is definitely not econometric in this sense as it involves first the fitting of a utility from raw data. However, if we will define econometrics to be the broader activity of fitting economic variables using data then we are definitely employing the econometric approach in the present work. The approach used in (Kocoska et al. (2009)) allows researchers to implement this data fitting procedure by using a very elegant non-linear optimisation algorithm to fit a set of parameters from which the elasticities may be easily deduced using the sensitivity analysis of associated linear programming problems. The purpose of this paper is to provide a theoretical underpinning for the utility fitting or estimation step in this procedure.

In practise we only have access to a finite selection of observed consumption data. We do not have access to the hypothesized underlying utility. The problem of fitting a utility function to this finite data set in a way that yields the same preference structure has already been solved by (Afriat (1967)). We observe here that one can fit a concave Afriat utility to any finite sample of consumption data taken from any (not even concavifiable) utility. This provides an approximation paradigm via the indifferent curves so generated.

The Afriat utility provides a well defined family of polyhedral indifference curves (irrespective of the arbitrary scaling issue of the utility itself). As we collect more data we may refine our approximation of these level curves and hence the question arises as to whether one can validly discuss some notion of convergence to a limiting preference structure as the data sampled tends to an infinite "dense" set of points. Indeed is it possible to have a convergence, in some well defined sense, to an underlying utility that rationalises this data set? It is this question we discuss in this paper and provide some concise theory for a positive answer to this question. The critical tool is the adaption of variation limits (Rockafellar and Wets (1998)) to this problem. The reason for the success on this mathematical convergence notion lies in its ability to characterise the set-convergence of indifference curves and to characterise the stability of optimal solutions under objective function perturbations.

The main contribution of this theory makes to the literature is that it provides a set of reasonable condition on the demand correspondence that ensures the existence of a utility that rationalises the preference structure and give rise to the observed demand (see Theorem 4). Moreover we can assert that the demand correspondence and the utility can be reconstructed, via the fitting of a sequence of Afriat utilities, using only a countable collection of observations. In this sense this theory provides an existence proof of an underlying utility. We can also give condition that ensure the fitted utility is actually concave improving on the results of (Kannai (2004)).

Keywords: *Economic modeling, utility function, revealed preferences, variational approximation*

1 SOME REVEALED PREFERENCE THEORY

In this section we provide a summary of some revealed preference theory that is relevant to the mathematical formulation of our theory on convergence.

The consumer is assumed to choose a consumption bundle within their budget $w > 0$ with respect to the associated prices p and so choose in $BG(p,w) := \left\{x \in \mathbf{R}^n \mid \langle p, x\rangle := \sum_{i=1}^n p^i x^i \le w\right\}$ an x that is at least preferred to all elements in $BG(p,w)$. As $BG(p,w) = B(\lambda p, \lambda w)$ for all $\lambda > 0$ we may redefine our unit of currency so that the whole budget has unit wealth with respect to the new currency i.e. $w = 1$. Having done this the prices actually correspond to dimensionless proportions. Thus a consumer is deemed to consume a commodity bundle $x \in X_{\mathcal{R}}(p)$ at a price p if it solves the utility maximization problem $x \in \arg\max\{u(x) \mid \langle p, x\rangle \le 1\}$, where the "arg max" refers to the elements that achieve the maximum in the optimization problem and the optimal value function $v(p) := \max\{u(x) \mid \langle p, x\rangle \le 1\}$ is referred to as the indirect utility. The indirect utility $v(p)$ assigns the maximum utility to a consumer for a bundle of goods under price p.

Usually the preferred set $\mathcal{R}(x) := \left\{y \in \mathbf{R}^n_+ \mid y\mathcal{R}x\right\} \equiv S_{-u}(x)$ is assumed convex and thus u must be a quasi–concave function, when it exists (or $-u$ is quasi–convex when $S_{-u}(x)$ are all convex). The function v is quasi-convex (i.e. $S_v(p)$ are all convex) and under mild assumptions (Martinez-Legaz (1991)) we have a dual formulas

$$v(p) = \sup\{u(x) \mid \langle x, p\rangle \le 1\} \quad \text{and} \quad u(x) = \inf\{v(p) \mid \langle x, p\rangle \le 1\}. \tag{1}$$

$$\text{and so} \quad \operatorname{Graph} X_{\mathcal{R}} = \{(p,x) \mid u(x) = v(p)\} \tag{2}$$

When this symmetric duality (1) holds we have the corresponding optimal solution set $P_{\mathcal{R}}(x)$ defined by $\{p \mid u(x) = v(p)\}$. On comparison with (2) we see that the graph of $\operatorname{Graph} P_{\mathcal{R}}$ corresponds to $\operatorname{Graph} X_{\mathcal{R}}^{-1}$. That is, $p \in P_{\mathcal{R}}(x)$ if and only if $x \in X_{\mathcal{R}}(p)$. In practise even when u is assumed continuous, we are not assured that v is finite everywhere (as the supremum may attain $+\infty$) but the function v is evenly quasiconvex (i.e. the sets $S_v(p)$ are the intersection of open half spaces).

We do not have a direct ability to observe $u(x)$ but $x \in X_u(p)$ where $X_u(p)$ stands for $X_{\mathcal{R}}$ when $\mathcal{R}$ is generated by a utility u. We say that $x \in X_{\mathcal{R}}(p)$ is a revealed preference to y and denote this by $x \succeq_{X_{\mathcal{R}}} y$ when $\langle p, x - y\rangle \ge 0$. That is y was in budget as $1 \ge \langle p, x\rangle \ge \langle p, y\rangle$ but as $(x,p) \in X_{\mathcal{R}}$ we have chosen x instead of y. That is we know that y in not in $\arg\max\{u(x) \mid \langle p, x\rangle \le 1\}$.

The transitive closure of $\succeq_{X_{\mathcal{R}}}$ gives a partial order $\succeq_R$ that denotes $x \succeq_R y$ when there exists $x = x_0$, x_1, $\ldots, x_n = y$ with $x_{i+1} \succeq_{X_{\mathcal{R}}} x_i$ for all i. If the preference is nonsatiated we denote $x \succ_R y$ when $x \succeq_R y$ and there exists i with $x_{i+1} \succ_{X_{\mathcal{R}}} x_i$ or $\langle p_{i+1}, x_{i+1} - x_i\rangle > 0$ for $(x_i, p_i) \in X_{\mathcal{R}}$.

More concisely let $\mathcal{X}$ denote configuration space of all samples $\{(x_i, p_i) \mid x_i \in X_{\mathcal{R}}(p_i)\}$. The *generalised axiom of revealed preference* (GARP) in its traditional form (Fostel et al. (2004)) says that there can not exists a cycle $\{(x_i, p_i) \mid i = 0, \ldots, q\} \subseteq \mathcal{X}$ with $x_0 = x_{q+1}$ and $\langle p_{i+1}, x_{i+1} - x_i\rangle \ge 0$ unless $\langle p_{i+1}, x_{i+1} - x_i\rangle = 0$. It is known that GARP is necessary and sufficient for the existence of a preference order $\succeq$ on $\mathcal{X}$ such that $x \succeq y$ whenever $x \succeq_R y$ and $x \succ y$ whenever $x \succ_R y$ (Kannai (2004)). That is there is an order $\succeq$ rationalizes $\mathcal{X}$.

A geometric characterisation of the demand functions exists when we know the utility function u (Eberhard and Crouzeix (2007)). Assuming the standard "non-satiation" assumption any optimal value x of (1) satisfies $\langle x, p\rangle = 1$ and if $u(x) = v(p)$ we have from (1) that if $\langle p', x\rangle \le 1 \Longrightarrow v(p') \ge v(p)$. Thus we may write

$$X_u(p) = \left\{x \in \mathbf{R}^n_+ \mid \langle x, p\rangle = 1 \text{ and } \langle p' - p, x\rangle \le 0 \text{ implies } v(p') \ge v(p)\right\}. \tag{3}$$

Alternatively, using the contrapositive we may denote

$$\tilde{S}_v(p) := \left\{p' \in \mathbf{R}^n_+ \mid v(p') < v(p)\right\}$$

$$\text{then} \quad x \in X_u(p) \quad \text{iff} \quad \langle x, p\rangle = 1 \text{ and } \forall p' \in \tilde{S}_v(p) \Longrightarrow \langle p' - p, -x\rangle < 0 \tag{4}$$

In (Eberhard and Crouzeix (2007)) we study functions $v : S \to \mathbf{R}$ such that closure of the (convex) strict level sets $\tilde{S}$ (as defined in (4)) satisfies $\overline{\tilde{S}_v(p)} = S_v(p) := \{p' \mid v(p') \le v(p)\}$ and for which $\operatorname{int} \tilde{S}_v(p) \ne \emptyset$ if $v(p) > \min v$. We call these solid pseudo–convex functions. For solid pseudo–convex function we have a characterisation

$$x \in [-N_v(p)] \cap \{x \mid \langle x, p\rangle = 1\} \equiv X_u(p), \tag{5}$$

where the normal cone to the level set $\tilde{S}_v(p)$ at p is defined as $N_v(p) := \left\{y \mid \langle p' - p, y\rangle \le 0 \quad \text{for all } p' \in \tilde{S}_v(p)\right\}$.

We see that the demand correspondence essentially supplies differential data regarding the orientation of the boundary of the level set (that is the indifference curves). We need to reconstruct these level curves from this data. For a finite data set this may be achieved using the Afriat procedure, which fits a concave utility or a convex indirect utility.

1.1 The SARP and Afriat Utility

Revealed preference theory contains a number of competing notions some of which have only recently been shown to be essentially equivalent (Eberhard et al. (2009a)). One that features very early in the history of revealed preference theory is the following: The strong axiom of revealed preferences (SARP) (Houthakker, 1950) holds if: for all $p_0, \ldots, p_q \in \mathbf{R}^n$ such that there exist $x_i \in X_u(p_i)$ with

$$\langle p_i, x_{i+1} - x_i \rangle \leq 0 \quad \text{for} \quad i = 0, \ldots, q-1$$

and $x_j \neq x_i$ for $i \neq j$ then we have $\langle p_q, x_q - x_0 \rangle \leq 0$ for all $x_q \in X_u(p_q) \setminus \{0\}$.

It is shown in (Eberhard et al. (2009a)) that GARP holds iff SARP holds. SARP is actually closely related to the concept of cyclical pseudo–monotonicity. A multifunction $\Gamma : D \rightrightarrows \mathbf{R}^n$ is called cyclically pseudo–monotone of order q if for all $i = 0, \ldots, q-1$ and $(p_i, x_i) \in \text{Graph } \Gamma$, with $x_i \neq 0$, we have

$$\langle p_i, x_{i+1} - x_i \rangle \geq 0 \quad \text{implies} \quad \langle p_q, x_0 - x_q \rangle \leq 0 \quad \text{for all} \quad x_q \in \Gamma(p_q). \tag{6}$$

A relation which is cyclically pseudo-monotone of all orders $q \geq 1$ is called cyclically pseudo-monotone (CPM) and maximally cyclically pseudo–monotone if its graph is not properly contained in the graph of any other cyclically pseudo monotone relation Δ with the same effective domain $\operatorname{dom} \Delta := \{x \in D \mid \Delta(x) \setminus \{0\} \neq \emptyset\} = \operatorname{dom} \Gamma$. A cyclically pseudo-monotone relation of order $q = 1$ is called pseudo–monotone (PM) with maximality defined analogously. Clearly SARP corresponds to $-X_u$ being CPM. When $-X_u$ is pseudo–monotone then X_u satisfies the classical weak axiom of revealed preference theory (WARP).

Definition 1 *Placing $I = \{1, \ldots, m\}$ let*

$$a_{ij} := \langle p_i, x_j - x_i \rangle \quad \text{for } i, j \in I \quad \text{and}$$
$$b_{ij} := \langle x_i, p_j - p_i \rangle \quad \text{for } i, j \in I$$

We refer to the following inequalities as the direct Afriat inequalities

$$\phi_j \leq \phi_i + \lambda_i a_{ij} \quad \text{for } i, j \in I. \tag{7}$$

We refer to the following inequalities as the indirect Afriat inequalities

$$\psi_j \geq \psi_i - \mu_i b_{ij} \quad \text{for } i, j \in I. \tag{8}$$

We note that SARP≡GARP holds for X_u iff there is a feasible solution to the direct Afriat inequalities (in (ϕ_i, λ_i) for $i, j \in I$, see (Fostel et al. (2004))). By the symmetric duality between the direct and indirect utility we also have SARP≡GARP holds for X_u iff there is a feasible solution to the indirect Afriat inequalities (in (ψ_i, λ_i) for $i, j \in I$.

Definition 2 *Given a set of data $(\{x_i, p_i\})_{i \in I}$ and a set of direct parameters $\{(\phi_i, \lambda_i)\}_{i \in I}$ we define the direct Afriat utility as:*

$$u(x) := \min \{\phi_1 + \lambda_1 \langle p_1, x - x_1 \rangle, \ldots, \phi_m + \lambda_m \langle p_m, x - x_m \rangle\}. \tag{9}$$

Given a set of data $(\{x_i, p_i\})_{i \in I}$ and a set of indirect parameters $\{(\psi_i, \mu_i)\}_{i \in I}$ we define an indirect Afriat utility as:

$$v(p) := \max \{\psi_1 - \mu_1 \langle x_1, p - p_1 \rangle, \ldots, \psi_m - \mu_m \langle x_m, p - p_m \rangle\}. \tag{10}$$

This brings us to the classical work of (Afriat (1967)) in the form given by (Fostel et al. (2004)). We have GARP necessary and sufficient for the ability to fit an Afriat utility that induces the same preference structure. When a strictly increasing, continuous function $k : \mathbf{R} \to \mathbf{R}$ exists such that $k \circ v$ is convex we say v admits a convex representation (and similarly for u admitting concave representation).

As GARP holds iff SARP holds and since any data set sampled from a valid solid, pseudo–concave utility function leads to a finite data set that satisfies GARP (Eberhard and Crouzeix (2007)) it follows that a concave Afriat utility may be fitted. A simple constructive algorithmic proof is given in (Crouzeix et al. (2009)). This is true even if the initial underlying pseudo–concave utility has no equivalent concave representation (such utilities do exist). From an approximation perspective this is an important observation we have made. Symmetrically a convex indirect utility may also be fitted to such a finite samples even when the original indirect utility is not convex nor has a convex representation.

1.2 Fitting the Afriat Utility to Data

As noted in the last section one may progressively fit an Afriat direct (indirect) utilities as the size of the data set grows despite the possibility that the underlying true utility is not itself concave (convex).

Here we sample increasingly larger data sets from a known Cobb-Douglas utility (plotted in the first figure) and sequentially fit an Afriat utility using the methods of (Kocoska et al. (2009)). One may form a best fit optimization problem U-NLP by using the Afriat inequalities plus the constraint $\lambda \geq 1$ with an objective $\min_{(\phi,\lambda)} \sum_i \lambda_i$. Once the best values for (ϕ, λ) are generated we may plot the level curves of the fitted Afriat utility. We generate a finite demand sample for a randomly generated set of prices. We plot in the following diagrams the commodities generated and the level curves of the fitted Afriat utilities for increasing sample sizes $k = 40$, 60. We do this for a commodity bundle of size two so that we may plot level set diagrams illustrating the convergence in action.

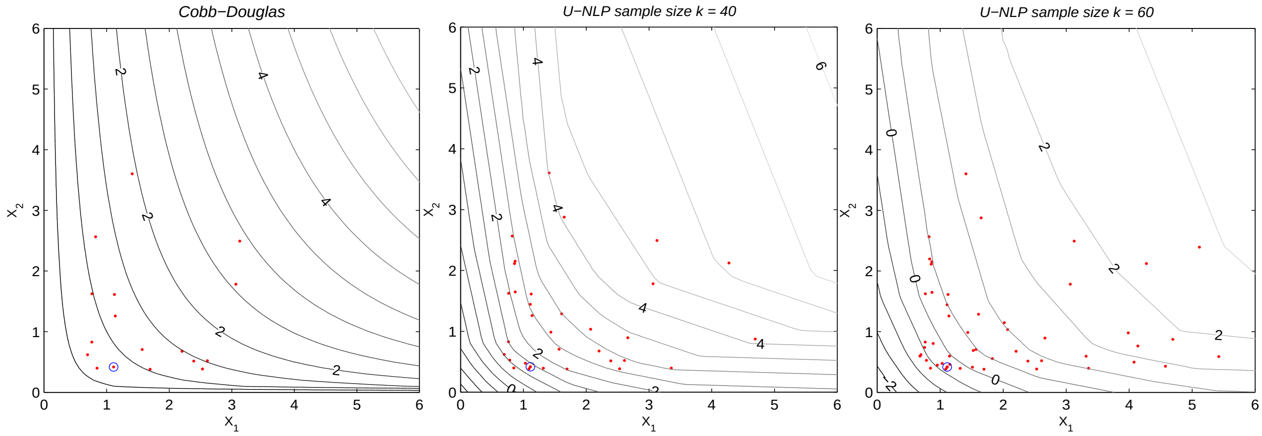

Figure 1. The original Cobb-Douglas and two Afriat utilities fitted to samples of size $k = 40$, 60.

2 CONVERGENCE NOTIONS AND MAIN RESULTS

As noted in last section we can always fit an Afriat utility to any finite set of data sampled from any solid, pseudo-concave utility. Thus for each data set $\{(x_i, p_i)\}_{i=1}^m$ we can obtain a set of convex, polyhedral, level curves of an indirect utility v_m specifying a normal cone $N_m(p)$ with that property that for each $i = 1, \ldots, m$ we have $x_i \in N_m(p_i)$.

In the following analysis we will have need to use a number of results taken from variational analysis. A sequence of extended-real-valued functions $\{f_m\}_{m=0}^{\infty}$ epi-converges to an extended-real-valued function f if both of the following hold:

$$e\text{-}\limsup_m f_m(x) := \min_{\{x_m \to x\}} \limsup_m f_m(x_m) \leq f(x) \quad \text{and} \tag{11}$$

$$e\text{-}\liminf_m f_m(x) := \min_{\{x_m \to x\}} \liminf_m f_m(x_m) \geq f(x). \tag{12}$$

The minimum is taken over all possible convergent sequences $x_m \to x$. There are many equivalent characterisation of epi-convergence (see Rockafellar and Wets (1998)) one being the set convergence of the epigraphs $\operatorname{epi} f_m := \{(x, \alpha) \mid \alpha \geq f_m(x)\}$ to the epigraph $\operatorname{epi} f$. The relevance of epi–convergence to utility approximation is brought into focus by the following fact: A sequence of functions epi-converges if and only if their level sets converges as sets. As level sets correspond to indifference curves this is exactly the behaviour we seek from such an approximation.

A sequence of set valued function $\{\Gamma_m\}_{m=1}^{\infty}$ graphically converges to Γ iff both of the following hold:

$$\Gamma(x) \subseteq \bigcup_{\{x_m \to x\}} \liminf_m \Gamma_m(x_m)$$

$$\text{where} \quad \liminf_{x_m \to x} \Gamma_m(x_m) := \{y \mid \exists y_m \in \Gamma(x_m) \text{ with } y_m \to y\} \quad \text{and}$$

$$\Gamma(x) \supseteq \bigcup_{\{x_m \to x\}} \limsup_m \Gamma_m(x_m)$$

$$\text{where} \quad \limsup_{x_m \to x} \Gamma_m(x_m) := \{y \mid \exists m_k \to \infty \text{ and } \exists y_{m_k} \in \Gamma(x_{m_k}) \text{ with } y_m \to y\}.$$

There exists theorems that link epi–convergence of a sequence of convex functions $\{f_m\}_{m=1}^{\infty}$ to f and graphical convergence of the subdifferential $\partial f_m(x) := \{z \in \mathbf{R}^n \mid f_m(y) - f_m(x) \geq \langle z, x\rangle$, for all $y \in \mathbf{R}^n\}$. Indeed epi–convergent sequences of convex functions must be yield a convex function f for which case we can also assert that the graphical limit of the subdifferential satisfies

$$\partial f(x) = g\text{-}\lim_{m\to\infty} \partial f_m(x).$$

We connect with our demand correspondence by observing that the normal cone to the level set $S_{f_m}(x)$ correspond to $N_{f_m}(x) = \operatorname{cone} \partial f_m(x) := \cup_{\lambda>0} \partial f_m(x)$. These observations allow a comprehensive treatment of the convex case to be carried out using known standard results from variational analysis. This we present in the next section.

2.1 The Case of a Concave Utility

What might happen as the sampling size increases? We can still fit a concave utility to each finite sample as the sample size m increase. Each time we obtain the data $\{(\phi_i^m, \lambda_i^m)\}$ and as before we have fitted $\phi_0 = u_0$ as a nominal fixed value. Is there some sense via which these approximate utilities converge to a function that can be rightfully called the underlying utility function. The next results gives some criteria for this to occur. In effect we provide here reasonable conditions that ensure convergence of the concave Afriat direct utilities to a concave direct utility function that may validly be associated with a indirect utility and therefore be deemed as a candidate for the underlying "real" utility function that rationalises the preference structure. These results improve on those of (Kannai (2004)) in that we provide the variational sense in which these approximations converge in both the primal and dual sense. The conditions given are merely sufficient for a desirable convergence. Recall that the relative interior $\operatorname{ri} C$, of a convex set C, corresponds to the interior relative to the affine hull of C. We denote the convex hull of set C by $\operatorname{co} C$. Denote the conjugate of a convex function f by $f^*(x) := \sup_y [\langle x, y\rangle - f(y)]$.

Theorem 3 ((Eberhard et al. 2009a)) *Suppose we have an increasing family of subsets*

$$S_m := \{(x_i, p_i),\ i = 1, \dots, m\} \subseteq \mathcal{X}$$

where $D_m := \{x_i \mid\ i = 1, \dots, m\}$ *with* $D := \cup_m D_m$ *dense in* $\overline{\operatorname{co}} D$, *with* $\operatorname{int} \overline{\operatorname{co}} D \neq \emptyset$, $0 \in \overline{\operatorname{co}} D$ *such that each* S_m *satisfies GARP. Suppose we have fitted the associated family of concave direct utilities* $\{u_m\}_{m=1}^{\infty}$ *with* $\phi_0 = u_0$ *(a nominal fixed value) and parameters* (ϕ_i^m, λ_i^m) *with* $\lambda_i \geq 1$ *for* $i = 1, \dots, m$. *Suppose in addition that the family of sets*

$$R_m := \operatorname{co}\{\lambda_i^m p_i \mid i = 1, \dots, m\},$$

are uniformly bounded and converging to a bounded set R *i.e.* $\limsup_m R_m = R$.

1. *Then* $\{u_m\}_{m=1}^{\infty}$ *epiconverges to a proper concave upper semi-continuous, utility function* u.
2. *Let* $v_m(p) := \max_y \{u_m(y) \mid \langle y, p\rangle \leq 1\}$ *and* $v(p) := \max_y \{u(y) \mid \langle y, p\rangle \leq 1\}$. *Suppose we have the following sufficient condition holding*

$$\begin{aligned} &\exists (\gamma, \mu) \in \mathbf{R}_{++} \times \left(S_1 \cap (l_1)_{++}\right) \quad \text{such that} \\ &\gamma p = \sum_i \mu_i \lambda_i p_i \quad \text{and} \quad \gamma p \in -\operatorname{ri} \operatorname{dom}(-u)^*, \end{aligned} \tag{13}$$

   where $S_1 := \left\{\{\mu_i\}_{i=1}^{\infty} \in (l_1)_+ := \{\{\mu_i\}_{i=1}^{\infty} \in l_1 \mid \mu_i \geq 0\} \mid \sum_i \mu_i = 1\right\}$ *and* $(l_1)_{++} := \{\{\mu_i\}_{i=1}^{\infty} \in l_1 \mid \mu_i > 0\}$. *Then* $\{v_m\}$ *converges pointwise to the associated indirect utility* v *on* $\mathbf{R}_+^n \setminus \{0\}$.
3. *If* $-u$ *is a proper, solid, pseudo-convex function then* u *has the property that* $x \in X_u(p)$ *for* $p \in \operatorname{cone} \overline{\operatorname{co}}\{p_i,\ i = 1, 2, \dots\} \cap \mathbf{R}_+^n$ *iff* $-p \in \operatorname{cone} \partial(-u)(x)$ *and* $\langle x, p\rangle = 1$ *for some* $x \in D$.
4. *In particular*

$$g\text{-}\limsup_m X_{u_m}(p) = X_u(p), \text{ if } p \in \operatorname{cone} \overline{\operatorname{co}}\{p_i,\ i = 1, 2, \dots\} \cap \mathbf{R}_+^n \setminus \{0\} \tag{14}$$

   and so if we have $x_m \to x$ *and* $p_m \in \operatorname{cone} \partial(-u_m)(x_m)$ *with* $p_m \to p \in \mathbf{R}_+^n \setminus \{0\}$ *and* $\langle x_m, p_m\rangle = 1$ *then* $x \in X_u(p)$.

2.2 The Case of a Pseudo-Convex Indirect Utility

When such conditions that ensure a concave limiting utility fail we can still assert that the sequence of fitted Afriat utilities provide us with a sequence of level curve families. This may be done by first defining for each m the strictly increasing, continuous function via the fitted indirect Afriat utility $v_m(p)$ i.e.

$$k_m(t) := v_m(p_1 t) \quad \text{where} \quad t > 0.$$

We then renormalise our Afriat utilities to produce another sequence of equivalent utilities

$$\hat{v}_m(p) := -k_m^{-1}(v_m(p))$$

which is the composition of a convex function on $\mathbf{R}^n$ and a concave increasing mapping on $\mathbf{R}$. Now p_1 lies on the level curve $\{p \mid \hat{v}_m(p) = -1\}$ for each m and also $\tau \mapsto \hat{v}_m(\tau p_1) = -\tau$ is finite. Note that $\operatorname{dom} \hat{v}_m = \mathbf{R}^n_+$ for all m and as v_m is a supremum function we have the convex subdifferential

$$\partial v_m(p) = \operatorname{co}\{-\mu_i x_i \mid v_m(p) = \psi_i^m - \mu_i^m \langle x_i, p - p_i \rangle\}.$$

As k_m strictly decreasing continuous then $-k_m^{-1}$ is strictly increasing and so the normal cone to the level set $S_m(\bar{p}) := \{p \mid \hat{v}_m(p) \le \hat{v}_m(\bar{p})\}$ is given by

$$N_m(\bar{p}) = \operatorname{cone} \partial \hat{v}_m(\bar{p}). \tag{15}$$

We may make the following change of origin and basis of the local coordinate system around p_1. Consider the direction $d = p_1 / \|p_1\|$ of strict monotonicity of $\hat{v}_m$ to be the nth vector in the canonical basis and p_1 the origin. Now a neighbourhood of $p_1$may be taken to have the form $V = Y \times T$ where Y and T are closed convex neighbourhoods of the origin in $\mathbf{R}^{n-1}$ and $\mathbf{R}$ respectively and the resultant function we will denote by $t \mapsto f_m(y,t)$ is decreasing. Set $\bar{\lambda} = \hat{v}_m(p_1) \equiv f_m(0,0)$. Let $\lambda_0 = \inf\{f_m(y,t) \mid (y,t) \in Y \times T\}$. For $\lambda > \lambda_0$ define

$$g_m(y,\lambda) = \inf\{t \mid f_m(y,t) \le \lambda\}, \qquad \lambda \in (\lambda_0, +\infty) \tag{16}$$

$$\text{for which} \quad N_{f_m}(y,t) = \operatorname{cone}\{(z,-1) \mid z \in \partial_y g_m(y,\lambda) \text{ for } \lambda = f_m(y,t)\}. \tag{17}$$

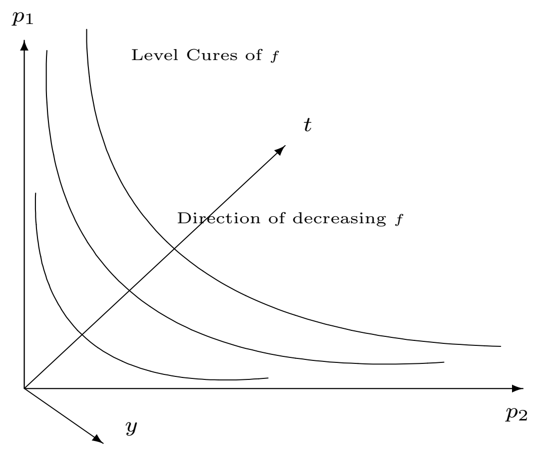

Figure 2. The new axis (y,t) are obtained by rotating the original axis for p.

Suppose f_m is lower semi–continuous in t and g_m defined as in (16) is continuous in λ then we have for $(y,t) \in Y \times T$ that

$$f_m(y,t) = \sup\{\lambda \mid g_m(y,\lambda) > t\}. \tag{18}$$

Roughly speaking, it can be said that a monotonic decreasing property in λ and a continuity property of $\lambda \mapsto g(y,\lambda)$ for any such family of proper, convex level set functions $\{g(\cdot,\lambda)\}_{\lambda \in \Lambda}$ correspond directly to a solid, pseudo–convex function f, as defined via the transformation (18), being strictly decreasing in t. Now suppose we have the epi-convergence of the convex functions $\{g_m(\cdot,\lambda)\}_{\lambda \in \Lambda}$. As $\operatorname{epi} g_m(\cdot,\lambda)$ corresponds to the indifference curve at level $\lambda = f_m(0,t) = -t$, convergence of $\operatorname{epi} g_m(\cdot,\lambda)$ corresponds to convergence of level curves, precisely the epi-convergence of $\{f_m\}_{m=1}^{\infty}$! Epi–convergence satisfies a compactness property: From any sequence of functions $\{g_m\}_{m=1}^{\infty}$ we may extract an epi-convergent subsequence and in this manner we may extract an epi–convergent subsequence from $\{f_m\}_{m=1}^{\infty}$. Now we may use graphical convergence of subdifferentials in (17). These considerations allow us to prove the following result that improves on (Crouzeix and Rapcsák (2005)) which treats only the case when f is differentiable. This theorem asserts the existence of a utility function based only on the existence of a suitable demand correspondence, giving a positive answer to the *problem of revealed preference.* The following is taken from (Eberhard et al. (2009b)) and to the authors knowledge this result is the only one of its kind in the literature.

Theorem 4 ((Eberhard et al. 2009b)) *Let $\Gamma(p) := -\operatorname{cone} X(p)$ where X is the demand correspondence. Suppose $\Gamma : D \rightrightarrows \mathbf{R}^n$ is cyclically pseudo–monotone (i.e. SARP holds for X) with closed graph and with convex, conic images on a closed, bounded set $D \subseteq \operatorname{dom}\Gamma$ such that $\overline{\operatorname{int} D} = D$. Suppose in addition there exists a $d \in \mathbf{R}^n$ such that $\langle x, d\rangle < 0$ for all $x \in \Gamma(p)\backslash\{0\}$ and $p \in D$. Then there exists a solid, pseudo-convex indirect utility function $v : D \to \overline{\mathbf{R}}$ such that $p \in \arg\min\{v(q) \mid \langle x, q\rangle \leq 1\}$ whenever $x \in X(p)$ and*

$$X(p) = -N_v(p) \cap \{x \mid \langle x, p\rangle \leq 1\} \quad \textit{for all } p \in \operatorname{int} D.$$

The proof is constructive in the sense that we approximate v via a subsequence of renormalised Afriat indirect utilities $\{\hat{v}_{m_k}\}_{k=1}^{\infty}$. Moreover the demand correspondence is approximated in the same fashion as given in (14) and we may assert Γ and N_v are simultaneously maximally pseudo- and cyclically pseudo–monotone. In particular if we only have access to a countably dense set of values $\mathcal{X} := \{(x_i, p_i)\}_{i=1}^{\infty} \subseteq \operatorname{Graph}X$ then we may write

$$X(p) = \left[\limsup_{\delta\downarrow 0} \operatorname{cone}\operatorname{co} X\left(B_\delta(p) \cap \{p_i\}_{i=1}^{\infty}\right)\right] \cap \{x \mid \langle x, p\rangle = 1\}.$$

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