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Why are simple models often appropriate in industrial mathematics?

de Hoog, F.R.

CSIRO Mathematical and Information Sciences, GPO Box 664, Canberra, ACT, 2601
Email: Frank.deHoog@csiro.au

Abstract: The application of mathematical modelling has been spectacularly successful in understanding, controlling and improving industrial processes, despite the fact that many industrial processes are quite complex. Furthermore, many successful models used to describe them are relatively simple. We examine this apparent contradiction through a number of case studies where simple models have captured the essence of processes we have encountered in industry and elsewhere.

A unifying feature is that industrial processes are often robust in the sense that they work effectively under a wide range of operating conditions. Often this is because key parts of the process tend to dominate. From a mathematical point of view, such processes are only weakly coupled to their environment, and a reductionist approach is often effective. In many cases, only a few key non-dimensional parameters associated with the underlying mathematical description dominate. Such problems are particularly amenable to simplifying mathematical analysis and approximation techniques.

Keywords: Mathematical Models, Process Models, Industrial Mathematics, Dimensional Analysis

1. INTRODUCTION

Throughout history, the development of mathematics has made substantial contributions to technology. For example, mathematics was important in the development of weapons such as the Greek catapult, the development of Galileo's telescope and the design of clocks. It is also the case that technology has been an important driver for the development of mathematics. The analysis of ballistics for example was important for the development of mathematics.

Arguably, the fabrication of canons and particularly the boring of barrels was one of the key drivers for the mathematics associated with process modelling. The heat produced during boring was one of the observations by Count Rumford that led to the first law of thermodynamics. It was also the motivation for Fourier's work on the use of trigonometric series to solve differential equations. However, it must have taken a substantial leap of faith to think that simple mathematical models would actually be useful. Almost all fabrication processes, including the boring of gun barrels, are quite complex and involve a variety of phenomenon, some of which may not be well understood. In the case of boring, there is turbulent flow of coolant and lubricant and a complex interaction of the tool with the work piece. For a more complex operation such as the fabrication of sheet metal in a modern rolling mill, the apparent complexity is immense. Nevertheless, the utilisation of mathematical models has been extremely successful when applied to this and other processes such as mineral separation and refining, metal casting and extrusion. Such models are often quite simple, particularly when compared to the apparent complexity of the process that they are addressing. The aim of the present paper is to explore this paradox.

This paper is organised as follows. Section 2 argues that while industrial processes may be complex, they often work because they are robust and that simple mathematical models are a direct reflection of this fact. The remainder of the paper illustrates the utility of simple models with three examples of increasing complexity. In the first example, dimensional analysis is used to demonstrate that considerable insight can often be obtained with very little effort. The second example, which arises in roll coating (painting), illustrates that dimensional analysis, when combined with some elementary analysis can often lead to dimensional reduction. The final example, from heat transfer in strip rolling, illustrates that relatively complicated models can often be substantially simplified

2. INDUSTRIAL PROCESSES ARE ROBUST

Despite the fact that most industrial processes appear to be complex, nature replicates a surprising number of these processes. For example observations of particles in the ripples in sand, caused by wind in the desert or by water at the beach, have demonstrated that particles have been segregation on the basis of size and specific gravity. Similar segregation takes place on a skree slope and in your favourite box of cereal. These are similar to the processes that take place in gravity separators such as jigs. Another example is crystal growth, which produces spectacular features such as stalagmites, stalactites and shawls in limestone caves. However, crystal growth is also a key process in the production of alumina, table salt and refined sugar. A further example is the regular cracking of mud and rock formations. Regular cracking is also important for producing coke with appropriate size distributions when volatilants are removed for coal in coke ovens.

The fact that many industrial processes share features that occur naturally suggest that many industrial processes are robust. Further support for this is that many industrial processes are quite old and were successfully implemented long before sophisticated computer control was even envisioned. It would simply not have been feasible in the past to implement processes that were not robust.

The question that now arises is "Can robust processes be described by simple mathematical models?" It turns out that often they can because they are weakly coupled to their environment. Then, only a few effects dominate and a reductionist approach is effective. Examples include:

- Rolling of Metal Strip
- Vibration of strings and shells
- Mixing of Dough
- Biosensor Devices
- Heat Transfer in Fuel Cells

Further simplification is often possible because of the existence of large and small parameters. Examples include:

- Lubrication (aspect ratios)
- Extrusion (aspect ratios)
- Coke Ovens (aspect ratios)
- Filtration (aspect ratios)
- Vibration of shells (aspect ratios, energy ratios)
- Stresses in coils (stiffness ratios)

3. DIMENSIONAL ANALYSIS

In his notebooks (da Vinci et al, 1954), Leonardo de Vinci wrote “Vitruvius says that small models are of no avail for ascertaining the effects of large ones; and I here propose to prove that this conclusion is a false one". He was referring to what we now think of as dimensional analysis. It is one of the most effective tools for the analysis of industrial processes and often provides substantial insight with very little effort.

Dimensional analysis is based on the fact that physically based phenomenon do not depend on the units chosen to describe their variables. For example, the physics of heating a work roll in a five stand mill is independent on whether we choose to describe the temperature in degrees Fahrenheit, Celsius or Kelvin. A key result is Buckingham’s Pi theorem which states that a physically meaningful equation that contains n physical parameters that can be described in terms of m fundamental physical quantities (length, mass, time, temperature, current), can be reduced to an equation that contains $n-m$ non-dimensional quantities. We will use two examples to exploit this; the first is the classical example of a oscillating pendulum while the second is of a jig used in gravity separation.

Consider the simple pendulum shown in figure 1. It seems plausible that there should be a relation between L the length of the pendulum, M the mass of the pendulum, T the period of the oscillation, g the gravitational acceleration and θ_0 the maximum angular displacement.

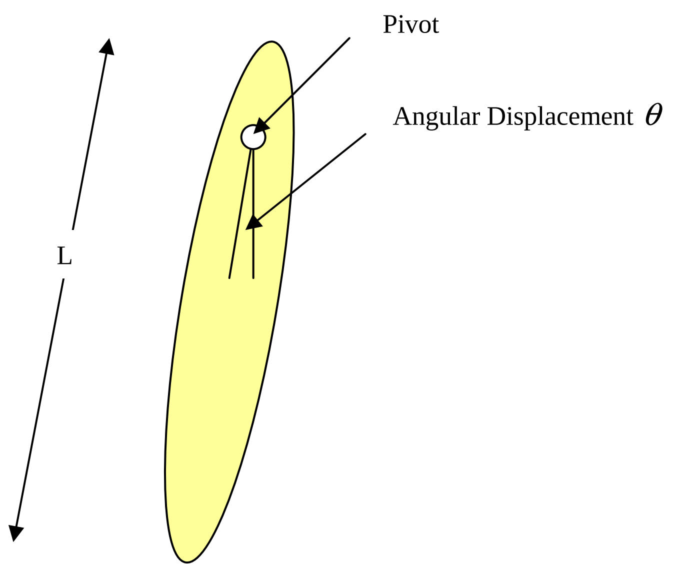

Figure 1: Schematic of simple Pendulum

For geometrically similar pendulums, we might expect a relationship of the form

$$f(L, M, T, g, \theta_0) = 0$$

which contains four dimensional physical parameters (L, M, T, g) that can be described in terms of three fundamental physical qualities (length, mass, time). By the Buckingham Pi theorem, only a single independent non-dimensional quantity is required in this situation. Consequently, we must have

$$F(\tau, \theta_0) = 0,\ \tau = T\sqrt{\frac{g}{L}}$$

or, equivalently

$$T = \sqrt{\frac{L}{g}} H(\theta_0)$$

for some (unknown) function H. Thus, the period of oscillation is proportional to the square root of the length L but does not depend on the mass M. For small displacements, $\theta_0 \approx 0$ and $H(\theta_0) \approx H(0)$, which establishes the well known experimental fact that the period of a pendulum depends only on its geometry. Although no explicit model has been formulated, quite a lot of potentially useful information has been obtained from this dimensional analysis. Even the effect of taking the pendulum to another planet has been quantified!

As a more challenging example, we consider the case of gravity separation using jigging (see for example Taggart, 1945). Figure 2 shows a simplified schematic of the jigging process.

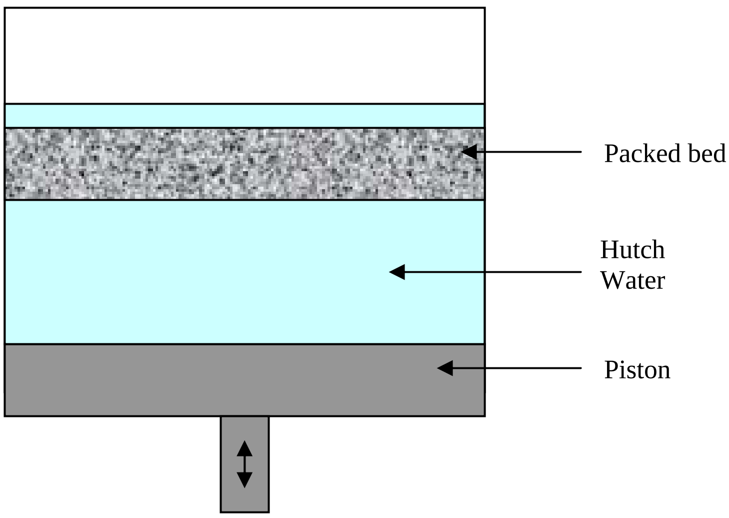

Figure2: Schematic of Jig

Essentially, a packed bed of granular material of different particle sizes and specific gravities sits on a screen and is flooded with water. A piston provides a pulse of water that briefly opens up the bed. Segregation of particles in terms of particle size and specific gravity takes place due to differential settling and the mobility of small particles (consolidation trickling) during the early stages of consolidation of the bed. While these processes are understood in a qualitative fashion, the detailed mechanisms are extremely complex. In reality,, a working jig also has a layer of relatively heavy particles that sits on the screen under the packed bed and there is a cross flow that transports the material from one side of the jig to the other. The heavy particles work their way through the packed bed, into the hutch water and the depleted ore (gangue) leaves the jig. Thus, the actual process is substantially more complicated than the schematic shown in figure 2.

Jigs and other mineral separation devices have been used and operated successfully for many years. Improvements to them have been made by placing them in a centrifuge, thereby increasing the effective gravitational field. Examples include the Kelsey centrifugal jig and the Falcon concentrator. We will use dimensional analysis to make an assessment of what changes are required for the operation of a jig when it is placed in a centrifuge.

The first task is to list the parameters that influence the operation of the jig. As with most complex processes, an exhaustive list would be quite long. For example, atmospheric pressure could influence cavitation near the piston for vigorous suction stroke and surface tension could impact recovery of fines. However the jig is one of the oldest separators used in mineral processing. It was used by ancient the Egyptians to separate material by placing them in woven baskets which were repeatedly lowered into water and then raised. Hence, there are good reasons for thinking that jigging is a robust process and that there is a relatively small number of variables that adequately describe the process. We assume that the key parameters are those given in table1.

Variable	**Description**	**Dimension**
L	Length scale for particles and the jig. Jigs and particles are assumed to be geometrically similar	Length
S	Stroke length	Length
T	Period of piston cycle	Time
ρ_w	Density of water	Mass/(Length)^3
ρ_p	Density of particles	Mass/(Length)^3
ν	Kinematic viscosity of water	(Length)^2/Time
g	Gravitational acceleration	Length/(Time)^3

Table 1: Dimensional Variables for Jigging

This gives seven dimensional variables whose dimensions are described in terms of three fundamental physical qualities (length, mass, time). Thus, there are four independent dimensionless groups. They are given in table 2.

Variable	Description
$SL/T\nu$	Reynold’s Number
$(S/T)/\sqrt{Lg}$	Froude Number
S/L	Length Ratio
ρ_w/ρ_p	Density Ratio

Table 2: Non-dimensional Variables for Jigging

Since the term S/T represents the order of magnitude for the velocity at the screen, the first two non-dimensional parameters can be interpreted as the Reynolds and Froude numbers respectively.

We can now infer feasible operating conditions for a jig in a centrifuge from the operation of the traditional jig. Given that the densities, and hence the density ratios are the same, we can ignore these variables. Let L, S, T and g denote the operating variables for the standard jig and let L_c, S_c, T_c and g_c be the variables for the centrifugal jig. Then, in order that the non-dimensional parameters remain unchanged, we require

$$L_c/L = S_c/S = (g_c/g)^{-1/3}, \quad T_c/T = (g_c/g)^{-2/3}$$

For example, if the effective gravitational acceleration in the centrifuge is $8g$, then halving the particle size (and other geometric quantities) and quadrupling the stroke rate should provide appropriate operating parameters for a centrifugal jig.

Of course, the dimensional analysis above is simply an abstraction and there are many differences between a standard jig and a centrifugal jig that do not scale appropriately. Nevertheless, the analysis does indicate that it is possible to perform mineral separation for smaller particles in a centrifugal jig than is possible in a traditional jig and this has been confirmed. In fact, the above scaling is remarkably accurate in practice when L is the length scale associated with the particles and the thickness of the packed bed, even when the length associated with other dimensions of the centrifugal jig do not scale.

4. SIMPLE ASYMPTOTICS AND SCALING

In the previous section, we examined dimensional analysis and showed how useful information can sometimes be obtained about a process just from knowledge of the relevant variables. However, while Buckingham’s Pi theorem just tells us how many independent non-dimensional parameters there are, it does not provide insight about the most appropriate non-dimensional groupings. Often, some simple preliminary analysis provides such insight as we show in the following example involving a rigid roll indenting “rubber” covered roll as shown in figure 3. The rigid roll is on an axle that can move in the vertical direction and the deformable roll in on a fixed axle. An analysis of this contact problem is part of a larger analysis of roll coating. Essentially the two roll rotate in opposite directions and the nip point is flooded with paint which induces a small gap to form between the rolls through which paint flows. The flow rate is controlled by the angular velocities of the rolls and the force applied between the rolls. Thus the cylinders act as a meter to control the volume of paint.

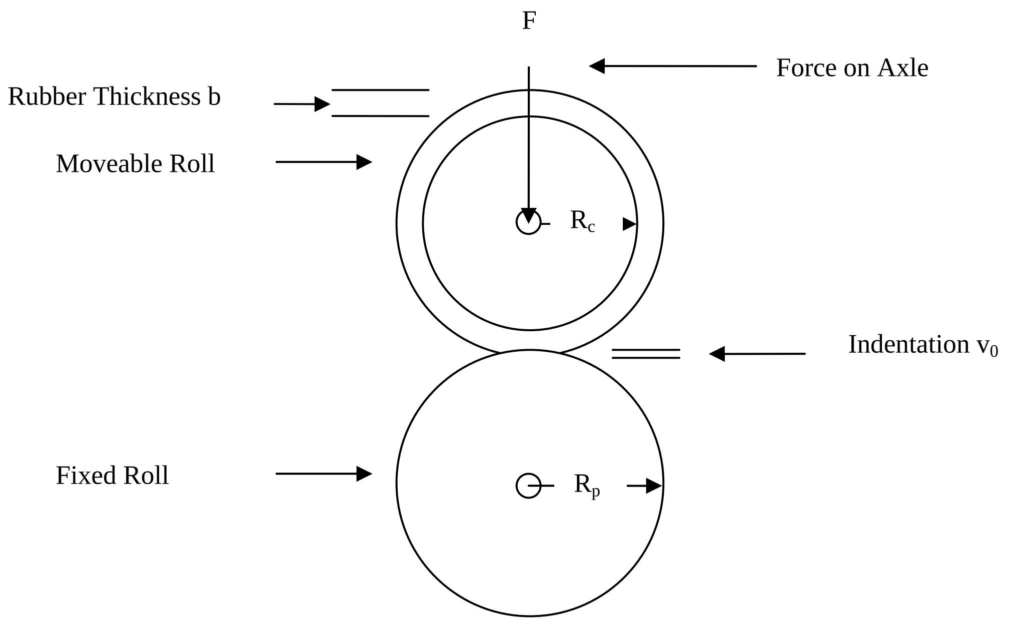

Figure 3: Rigid fixed roll indenting moveable rubber covered roll

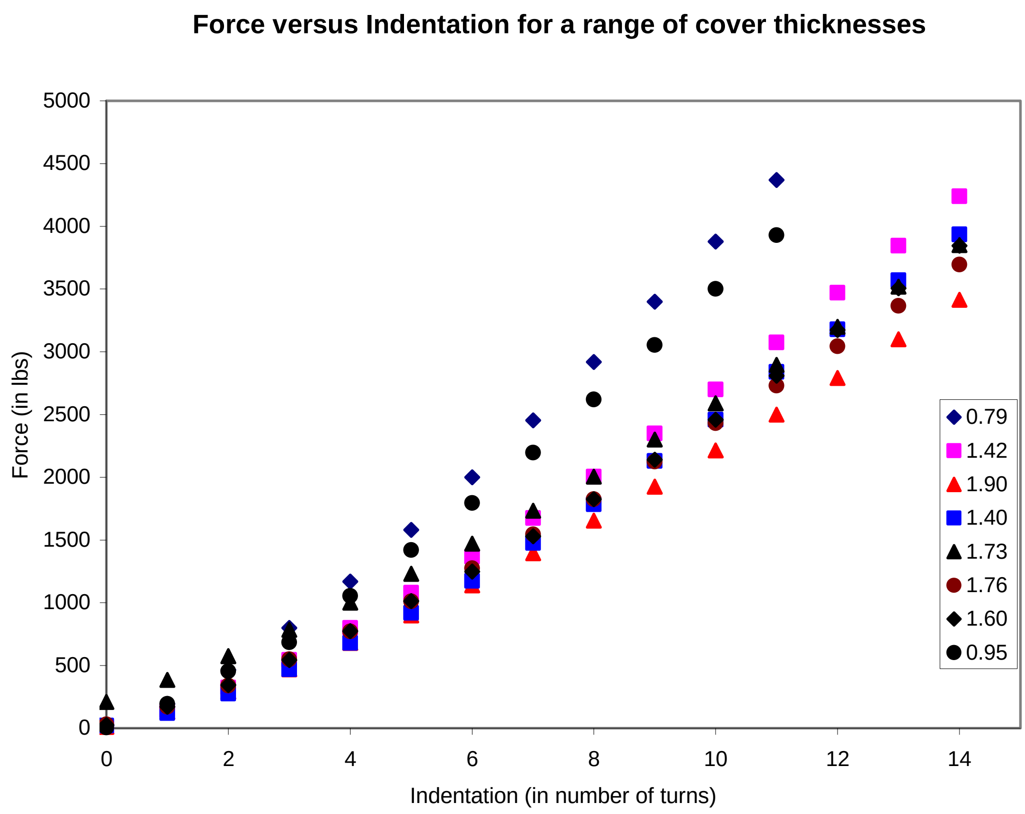

Figure 4: Plots of Indentation versus Force

A key issue is that, while the radii of the fixed roll and the rigid core of the movable rolls remain fixed, the thickness of the rubber on the moveable roll varies. The reason for this is that the rolls are ground on a regular basis to repair damage to the rubber surface sustained during use.

An experiment was performed where the rubber covered roll was indented. For each indentation, the force between the two rolls was noted and the experiment was repeated for a number of rolls with different rubber thickness. The results are shown in figure 4. It can be seen that for a number of the experiments there is a force between the rolls even when the indentation is supposedly zero. The reason for this is that the experiment was performed with the rolls rotating and with paint on the rolls. This made it difficult to judge when the rolls were just touching each other.

We will now demonstrate how dimensional analysis and some simple modelling can be applied to simplify the results of this experiment. The variables that are relevant for this problem are given in Table 3.

Variable	**Description**	**Dimension**
R_c	Radius of rigid core in top roll	Length
R_p	Radius of pickup roll (bottom roll)	Length
b	Thickness of rubber on top roll	Length
W	Width of roll	Length
v_0	Indentation	Length
F	Force between rolls	Mass Length/(Time)^2
E	Elastic modulus of rubber	Mass /((Time)^2 Length)

Table 3: Dimensional Variables for Indentation

Consequently, we have seven parameters whose dimensions are a combination of length, mass and time. However, only the inter-roll force F and the elastic modulus E contain dimensions mass and time. These can only be eliminated by taking the ratio F / E, which has the dimension of (length)^2. Thus, effectively, we have six parameters whose dimensions are length or length squared and we can therefore form five independent non-dimensional variables. Further more, because the only quantities that vary in the experiment outlined above are thickness of the rubber b, the indentation v_0 and the force F, we can choose the non-dimensional parameters so that at most three of them will vary.

To make further progress, additional information is required. Our starting point is to note that the indentation is small compared to the roll radii and the thickness of the rubber. Because the indentation problem depends only on the stress and strains in the neighbourhood, it is asymptotically equivalent to the indentation of an infinite strip of rubber on a rigid foundation that is indented by a roll whose radius is the harmonic mean of the top and bottom rolls. That is, a roll of radius

$$R = 2R_p(R_c + b)/(R_p + R_c + b)$$

as shown in figure 5.

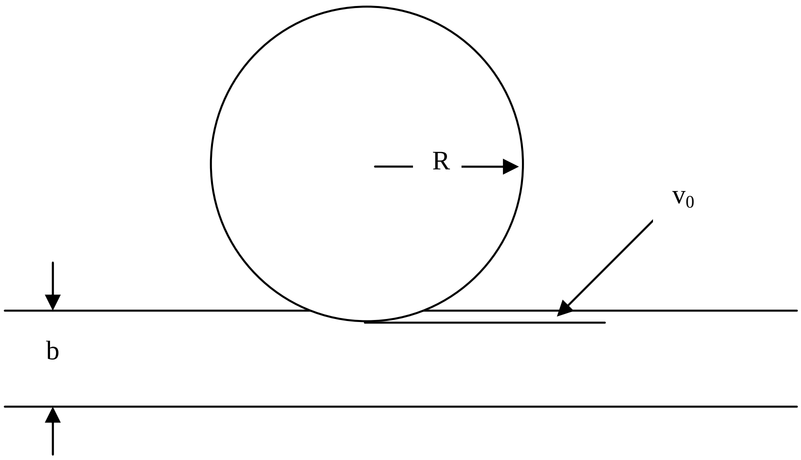

Figure 5: Asymptotically equivalent problem of cylinder indenting elastic layer

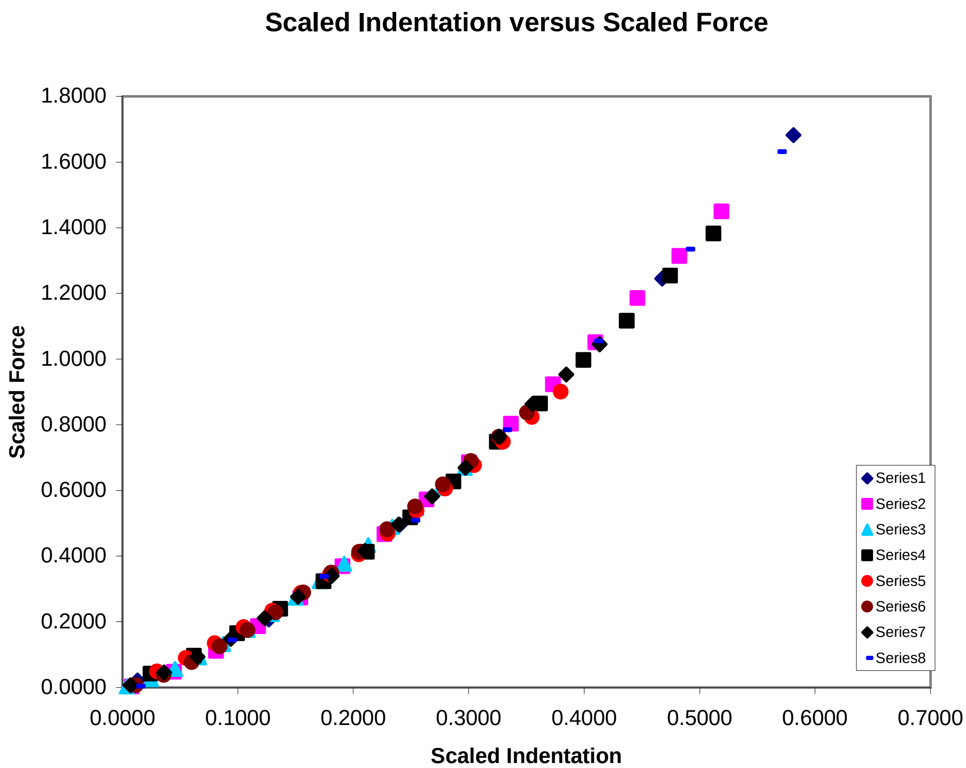

Figure 6: Plot of Scaled Indentation versus Scaled Force

A natural scaling for the spatial co-ordinates x and y is b and we therefore define the scaled variables $\bar{x} = x/b$ and $\bar{y} = y/b$. To leading order, the vertical displacement at the contact area is given by $v(x) \approx v_0 + x^2/2R$. On introducing the scaled displacements $\bar{v} = Rv/b^2$ and $\bar{u} = Ru/b^2$ we find that the scaled vertical displacement takes the simple form $\bar{v}(\bar{x}) = \bar{v}_0 + \bar{x}^2/2$, $\bar{v}_0 = Rv_0/b^2$. This suggests the scaling $\bar{\varepsilon} = R\varepsilon/b$ for the strains, $\bar{\sigma} = (R/Eb)\sigma$ for the stresses and the force on the axle as $\bar{F} = RF/EWb^2$ where E is the Young's modulus and W is the width of the rolls.

The simple analysis above suggests that the key non-dimensional parameters are the non-dimensional indentation $\bar{v}_0 = Rv_0/b^2$ and the non-dimension force $\bar{F} = RF/EWb^2$. After recalibrating the initial contact point, these quantities are plotted in figure 6 for the data in figure 4. The fact that all the data now appears to lie on the same curve supports the hypothesis that that there are only two important dimensionless variables.

5. SIMPLIFICATION OF MODELS

In rolling of steel and aluminium, the metal strip is deformed plastically by twin sets of rolls. These are called work rolls and they are in contact with large rolls, called backup rolls, to stop them bending. During rolling, residual stresses are induced in the sheet and this plays an important role in the "flatness" of the final product. Furthermore, the variation in the diameter of the work roll plays an important role in the formation of the residual stresses and hence the flatness of the product. Crown ground into the roll and thermal expansion are the two main components of variation in work roll diameter variation. Although ground in crown is fixed, thermal expansion of the work roll can be controlled to a certain extent by stray cooling.

Essentially, the work roll is heated by the work piece at the roll bite and is cooled by spray cooling, air cooling and by contact with the backup roll. This is shown schematically in figure 7. In order to effectively control the thermal expansion, it is necessary to efficiently calculate the temperature distribution of the work roll.

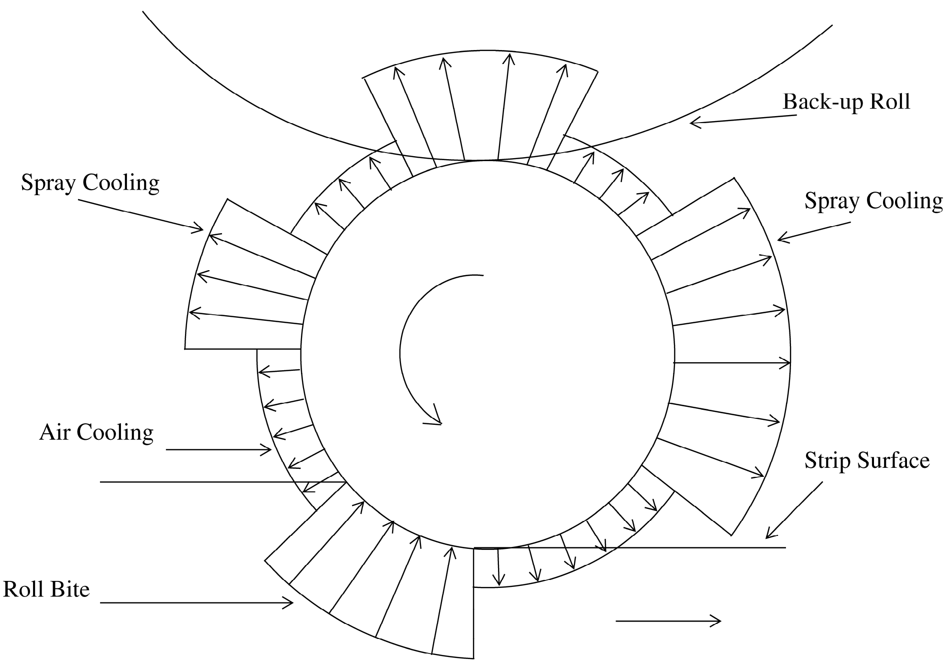

Figure 7: Heating and cooling of work roll

The time scale associated with the roll rotation is small compared with diffusion of heat through the roll. As a result, the temperature distribution in the roll has a thin layer at the surface of the roll where thermal gradients are large. Although the temperature in this layer does not contribute substantially to the thermal expansion of the roll, it is still an important part of the overall heat balance as heat transfer into the roll is determined at the boundary,. Thus, it would appear that a numerical scheme applied to this problem needs to resolve the layer as it plays a crucial role in the heat transfer from the perimeter to the interior. As the problem is three dimensional and time dependent, a numerical simulation of this complexity is not feasible for model based control. However, it turns out that substantial simplification is possible.

For simplicity, we restrict attention to the two dimensional case which exhibits all of the features of the more complex situation. For the extension to the three dimensional case, see (Robinson and de Hoog, 1996). The equation for heat conduction, in non-dimensional form, is

$$\frac{\partial T}{\partial t}+\frac{1}{\varepsilon^2}\frac{\partial T}{\partial \theta}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \qquad (1)$$

and it satisfies the boundary condition

$$\frac{\partial T}{\partial r}=-hT+q,\ r=1 \qquad (2)$$

and the initial condition

$$T(r,0)=0\,.$$

Here h, the heat transfer coefficient, and q, the heat flux at the roll bite, depend only on θ. However, subject to some limitations, the analysis can be extended to the case when h and q also have temporal variation.

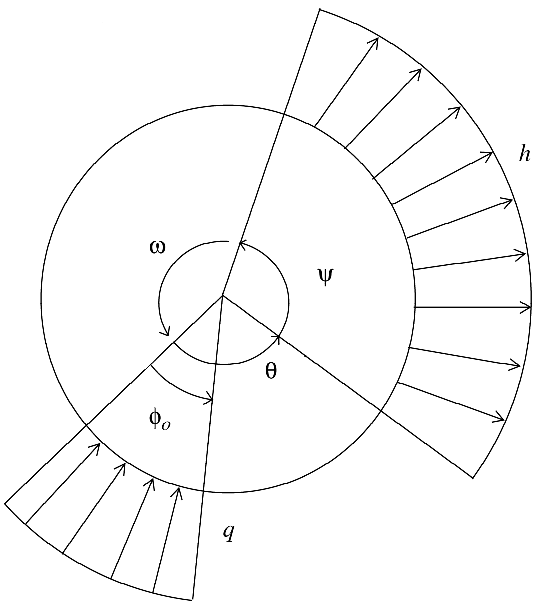

Figure 8: Piecewise constant h and q used as a standard test

Because heat transfer by advection (the rotation of the roll) is much faster than diffusion, the feature that dominates heat transfer at the circumference is the small parameter ε (ε^{-2} is the Peclet number). Essentially, the solution has the structure

$$T = \hat{T} + u + v \tag{3}$$

where $\hat{T}$ has no azimuthal dependence and is smooth near the circumference. The addition terms correspond to a transient term u which becomes negligible after a couple of revolutions and a layer term v which has no temporal component and is small except for a layer of order ε on the circumference. When h and q are piecewise constant as illustrated in figure 8 a typical temperature profile for the steady state case is given in figure 9. As noted previously, the outer layer decreases rapidly.

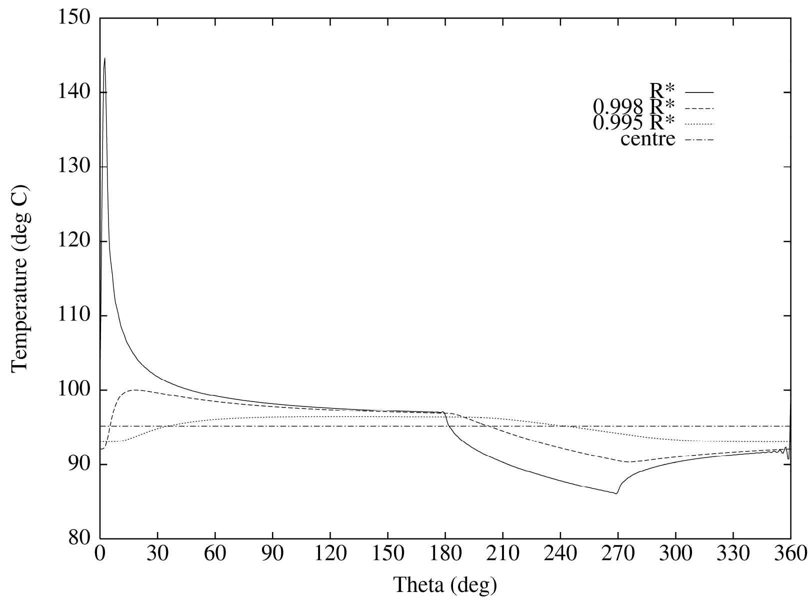

Figure 9: Temperatures in layer for a typical test problem

After a few revolutions (required to establish the boundary layer), we have

$$T \approx \overline{T} + v \tag{4}$$

where

$$\overline{T} = \frac{1}{2\pi}\int_0^{2\pi} T d\theta \,.$$

To leading order,

$$\frac{\partial \overline{T}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{T}}{\partial r}\right) \tag{5}$$

$$\frac{1}{\varepsilon^2}\frac{\partial v}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \tag{6}$$

Superficially, this provides a considerable simplification as the equation satisfied by $\overline{T}$ is reduced by one spatial dimension and v satisfies the steady state equation. However, we need to impose appropriate boundary conditions. On substituting (4) into (2) we obtain

$$\frac{\partial v}{\partial r} = -hv + Q \tag{7}$$

$$Q = -\frac{\partial \overline{T}}{\partial r} - h\overline{T} + q \tag{8}$$

As the problem is linear, we can write

$$v(r,\theta) = \int_0^{2\pi} K(r,\theta,\phi) Q(\phi) d\phi$$

and since the azimuthal average of v is zero, it follows that

$$\int_0^{2\pi} w(r,\phi) Q(\phi) d\phi = 0 \tag{9}$$

where

$$w(r,\phi) = \int_0^{2\pi} K(r,\theta,\phi) d\theta$$

We now define a scaled weighting function

$$w_1(\phi) = w(1,\phi) / \int_0^{2\pi} w(1,\phi) d\phi \tag{10}$$

On combining (8), (9) and (10) , we obtain

$$\frac{\partial \overline{T}}{\partial r} = -\hat{h}\overline{T} + \hat{q} \tag{11}$$

where

$$\hat{h} = \int_0^{2\pi} w_1(\phi) h(\phi) d\phi$$

$$\hat{q} = \int_0^{2\pi} w_1(\phi) q(\phi) d\phi$$

Now (5) subject to the boundary condition (11) would provide a very convenient way of calculating the azimuthally averaged temperature which is the quantity of interest when calculating the thermal expansion of the roll. The catch is that we don't know $\hat{q}$ and $\hat{h}$ (or equivalently the function w_1). However, from the definition of w and K, it turns out that w is the solution of

$$-\frac{1}{\varepsilon^2}\frac{\partial w}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2} \tag{12}$$

subject to the boundary condition

$$\frac{\partial w}{\partial r} = -hw + 1, \quad r = 1 \tag{13}$$

Equations (12), (13) are equivalent to a steady state problem with a uniform heat source on the boundary of a roll that is rotating in a clockwise direction. Thus, w and hence w_1 can be calculated in a relatively straightforward manner. A plot of w_1 for the problem illustrated in figure 8 is given in figure 10

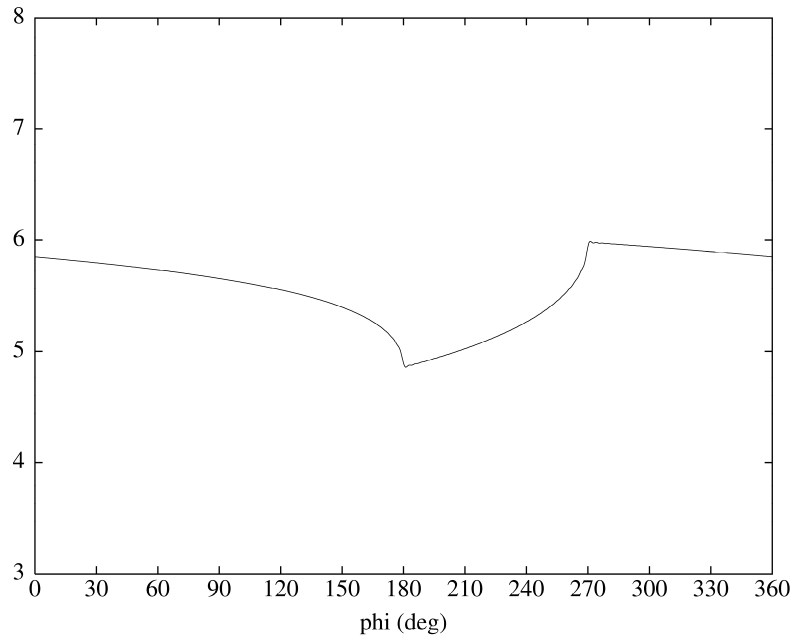

Figure 10: Unscaled weighting function $w(\phi)$

The approach outlined above has a number of attractive features. They include;

- the weighting function calculated via (12) and (13) is all that is required to analyse the effect of moving the sprays nearer to or further from the roll bite
- the boundary layer is removed from the calculation
- the spatial dimension of the problem is reduced by 1.

6. CLOSING REMARKS

Despite apparent complexity in manufacturing processes, simple mathematical models often provide remarkable insight. The reason for this is that many industrial processes are robust and this manifests itself in weak coupling between the process and its environment and in the existence of large/small non-dimensional parameters. Even when detailed understanding of the processes are lacking, dimensional analysis can provide useful information. Dimensional analysis coupled with partial model information can lead to dimensional reduction. When a model is completely specified, the existence of large and/or small parameters often provides an opportunity for model simplification.

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