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Container packing problem for stochastic inventory and optimal ordering through integer programming

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Abstract: A supplier to a major manufacturing company is trying to cope with a probabilistic inventory management problem that has stochasticity in demand, material consumption rate, delivery times and other logistics-related parameters. Based on the specialized nature of the finished goods, the input materials are imported from Asia and carried in refrigerated containers that are subject to size and weight limits. The material should be used within a few weeks of receipt; otherwise the rate of manufacturing faults increases drastically. Unsatisfied demand (stockout) can be recovered expensively through the use of air transportation. The stochastic inventory problem with variable demand and lead time along with the optimal packing of containers are modelled as a stochastic optimisation problem. The solution to this problem expresses the optimal order size for raw materials considering all the stochasticities and other logistic limitations by minimizing the expected cost of stockout and over-supply. In addition, the optimal packing of pallets of raw materials into the containers is calculated simultaneously. In this paper, along with the modeling of the probabilistic inventory problem, we develop integer programming models for the optimal container loading and compare the numerical outputs.

Keywords: *Integer programming, stochastic optimisation, container packing, optimal ordering, stochastic in-ventory*

1 INTRODUCTION

In this paper we are dealing with a stochastic inventory control problem. From one perspective, it could be considered as a typical inventory problem. However, there are some peculiarities so that the problem is more difficult to solve. There is a manufacturing company with highly volatile demand for final product, and therefore a highly volatile demand for raw materials. At the same time, the raw materials should be used within few weeks of their arrivals, otherwise, there is a defect cost regarding aging and perishability of the raw materials. In addition, raw materials are imported from overseas in particular refrigerated containers in which the pallets of raw materials should be packed in a particular order. We develop a stochastic optimisation model to find out optimal order quantity in the presence of volatile demand and variable lead time, the optimal container packing to carry all the demand and maximize volume utility of the containers.

Models for stochastic inventory control problem have attracted the attention of many researchers and there exists enormous amount of related literature. For this short paper, we do not intend to give an extensive literature review. However, some related publications will be referenced. Among thousands of publications in this area, Janakiraman and Roundy (2004) explain lost-sale problem with stochastic leading time. In this paper, authors considered stochastic demand and stochastic lead time to derive some analytical expressions for the cost function and solutions. Generally in the existence of stochasticity in demand and lead time, there should be some safety stock to mitigate the problem of stockout. Orcun et al. (2007) tried to determine safety stock size in the context of supply chain by means of some simulation environments. With normality assumption on random variables distribution of both demand and lead time, the order size could be stated as

$$Q^* = \mu_L \mu_D + z_\alpha \sqrt{\mu_L \sigma_D^2 + \sigma_L^2 \sigma_D^2}, \tag{1}$$

where μ and σ are mean and standard deviation for demand and lead time, and z_α denotes the number of standard deviations required to provide a service level of α, or, equivalently, a stockout probability of $1 - \alpha$. In this formulation, the parameter α creates the possibility of "what-if" scenarios for managerial practice. To obtain the formula (1) the assumption of normality is critical. However, there are other studies, e.g. Eppen and Martin (1988), about the optimal order size without normality assumptions. Also, Deng et al. (2010) investigated the periodic review stochastic inventory control problem with stochastic demand and lead time. The goal was to coordinate a sequence of orders of a single commodity, aiming to supply stochastic demands over a discrete finite horizon with minimum expected costs. By means of simulated annealing algorithm, they validate their analytical formulation for the problem. Rossi et al. (2012) addressed the general multi-period production/inventory problem with non-stationary stochastic demand and supplier lead time under service-level constraints. They proposed two hybrid algorithms that combine constraint programming and local search for computing near-optimal policy parameters.

In this paper, cutting and packing problem is considered in junction with stochastic inventory control. The problem is a well-motivated research issue in Combinatorial Optimization and Operations Research areas. From theoretical perspective, it is a NP-hard problem and is very difficult to solve. In addition, it has a wide variety of applications in the real world. As an example, the efficient cargo transportation is of high economic value in logistics, and optimal cargo loading leads to lower costs all over the distribution system. Dyckhoff (1990) extends a great typography about different versions of packing problem under different names. Chen et al. (1995) analytically formulated a container loading problem as a zero-one mixed integer programming problem, and they presented some numerical results for some different instances of this class. Junqueira et al. (2012) proposed a more realistic mathematical model in terms of mixed integer programming problem model for loading rectangular boxes into containers considering vertical and horizontal stabilities.

The loading problem which is considered in this paper has some specific attributes. Containers have three rows, and at most nine pallets can be loaded. Among four different kinds of pallets, they have two different categories. Pallets from the same category could be placed on top of each other in a row, but pallets from different categories cannot. Furthermore, there is a weight limit, floor limit, and volume limit for each container. The aim of this paper is to formulate probabilistic inventory problem with container loading as a stochastic optimisation problem, and solve the packing problem in this context. The rest of this paper is organised as follows. First a statement of the problem is presented. Then, a stochastic optimisation model is created. Our solution method is presented afterward. At the end, we explain the numerical experiments and conclusion sections.

2 Statement of the problem

A supplier to a major manufacturing company (Company) interested in a decision support tool to assist them in monthly order quantity considering existing uncertainties in different aspects of manufacturing including demand, delivery, faulty parts, perishability and so forth. The supplier has to establish a balance between avoiding stockout of materials, and the need to consume the materials within a few weeks of receipt due to deterioration in the state of material over time.

There are four different parts to be produced, and one manufacturing line on which the parts are manufactured. Materials to make the parts are in different sizes (thickness, width, length, weight and so forth) and are not interchangeable between parts. Because there is only one press line in the manufacturing process, tool change durations on the press line mean that Company avoids small lot sizes, and will instead tend to produce more than a day's demand for parts in one lot. In addition, there is a small inventory for completed parts which its size is decided by Company. The material should be used promptly (within six weeks), as aging effects reduce the materials suitability in the manufacturing process.

Company places orders on the supplier approximately 24 hours in advance. Demand for the final product varies from day to day, and is not accurately predictable. Also, the ratio of demand for one part to another varies due to the range of parts relates to more than one model of final products. Considering the fact that transit time is about 35 days, then it is assumed that typically the order decision is made, and the new order is placed, before the arrival of the previous month's order at the plant. Thus, there are two orders in transit during part of month.

Based on what has been explained so far, there are three major sources of uncertainty and variability that matter greatly in relation to the raw material inventory management:

- Uncertainty and variability on the monthly orders arrivals which is 35 days $\pm$ few days;
- Variability in the consumption rate of material, given a product mix ($\pm$ 5 %);
- Uncertainty and variability in the demand from Company, on a day to day basis as well as in aggregate over the approximately 35 days in between receipt of materials. The variability is quite high for this item and is possibly up to 20% compared to forecasts.

The critical decision for the supplier is the amount of materials to order at each monthly order point. They have to balance the need to avoiding stockout, and the need to consume raw materials within a few weeks of receipt. The decision is driven by the following data, and an optimal order quantity needs to be calculated considering all the uncertainties.

- Constant parameters, such as number of pallets in the system;
- Data on the probability distribution for transit times, regarding observed (historical) transit times;
- Data on the uncertainty in the consumption rate and faulty items which is increasing as materials becoming older;
- Data on estimates of day-to-day variation as well as weekly or monthly variation in the demand for the final products.

Along with the inventory problem, Company is interested to find optimal packing for containers to transit the raw materials into the manufacturing unit. The materials are transported in specialised pallets, which are stacked in temperature controlled shipping containers. The number of pallets per container depends on the size of the materials and therefore on the intended parts, however, seven or eight pallets per container is normal. Each container has its own weight and volume limit, and the pallets should be loaded in a particular order.

3 An inventory control problem with stochastic lead time and stochastic demand

The problem of optimal order size of raw materials for the above-mentioned manufacturing company is a periodic review stochastic inventory model. The purpose of an inventory control problem is to determine the rules that managers can follow to minimize the total cost of inventory maintenance and satisfying demand. In this problem, demand over a given time period is uncertain, and there is some uncertainty in the length of leading time, that leads to two particular kinds of cost here: The cost of unsatisfied demand, and the cost of perished raw materials. The unsatisfied demand has to be satisfied by air transit shipment of raw material, which is very expensive to Company. In addition, raw materials have to be used within six weeks of their arrival. Otherwise, ageing will change the molecular structure, and the materials will be vulnerable in manufacturing causing faulty parts.

The most significant sources of uncertainties are unpredictable demand and raw material transit time. Stochastic demand introduces the highest uncertainty in this problem. The demand for the final product could vary around 20%. The nature of market is highly volatile, and Company could have different unexpected promotional programs based on the rivals' behaviour in the market. In addition, the transit of material from overseas could take 35 ± 5 days. Company and the supplier have safe stock inventories to cope with late arrivals. However, the perishability nature of raw materials creates a hard decision making situation for the managers of both parties. Furthermore, the parts made of aged raw materials have variable level of faultiness which is approximated by 5%. The level of imperfect parts could increase as the holding time approaches to six weeks.

3.1 Stochastic optimisation model of the inventory problem

Company is interested to find out monthly order quantity size for four different types of raw materials, namely n_1, n_2, n_3 and n_4. In case the demand d_i is known for raw material i, considering that c_i is the order cost, b_i is the air transport cost ($b_i >> c_i$) and h_i is the holding cost, by solving the following deterministic problem, the optimal order quantity can be found:

$$\min_{n_i > 0} \max\{(c_i - b_i)n_i + b_i d_i, (c_i + h_i)n_i - h_i d_i\}. \tag{2}$$

It is understood from the model that there will be air transport cost if $n_i < d_i$, and holding cost if $n_i > d_i$. The objective function of problem (2) is piecewise linear, and the minimum is attained at $x_i^* = d_i$. It means if Company knows the demand beforehand, the optimal ordering size is the amount of the demand. However, the demand is a random variable D, and we assume that its cumulative distribution function (cdf) is known based on historical data, and is finitely supported. To imitate the stochastic model, we consider several scenarios for the demand of raw material i as $d_{i1}, \ldots, d_{iK}$ with the respective probabilities p_{ik}, where $\sum p_{ik} = 1$. Consider $v_{ik} = \max\{(c_i - b_i)n_i + b_i d_{ik}, (c_i + h_i)n_i - h_i d_{ik}\} \quad \forall i, k$. To find the optimal order size, we need to solve the following deterministic linear programming problem:

$$\begin{cases} \min & \sum_{i=1}^{4} \sum_{k=1}^{K} p_{ik} v_{ik} + \sum_{i=1}^{4} h_i I_i + \sum_{j=1}^{P} (x_j + \text{P[j].empty}) & \\ s.t. & v_{ik} \geq (c_i - b_i)n_i + b_i d_{ik} & \forall i, k, \\ & v_{ik} \geq (c_i + h_i)n_i - h_i d_{ik} & \forall i, k, \\ & I_i \geq (5 + \delta)\bar{d}_i & \forall i, \\ & \sum_{j=1}^{P} \text{P[j].patInf[i]} x_j \geq n_i & \forall i \in \{1, 2, 3, 4\}, \\ & n_i \geq 0 \ \forall i, \quad v_{ik} \text{ unrestricted } \ \forall i, k, \quad I_i \geq 0, \ \forall i, & \\ & x_j \text{ integer } \forall j. & \end{cases} \tag{3}$$

It is important to note that we have not considered any explicit perishability cost in this model. However, it is hidden implicitly in the holding cost. For an item, we can assume that deterioration starts from the arrival day. Therefore, the perishability is increases everyday, and we can assign a time-dependant penalty. As there is a six week period for deterioration (or 42 days), we can incorporate perishability penalty in terms of $M \times (e^{\frac{t}{42-t}} - 1) + h$. So far we only explained the stochasticity in the demand. Based on the description of the problem, the lead time takes 35 $\pm\delta$ days. Therefore, the latest time of delivery is $5 + \delta$ days after the beginning of the month. Also, consider that the existing inventory at the beginning of the next month is I, which could be positive, zero or negative. The $5 + \delta$ day period of the beginning of the month is perhaps the hardest period for Company to cope with slack or surplus inventory. If we consider the maximum daily demand in a particular month is $\bar{d}$, then by making $I \geq (5 + \delta)\bar{d}$ for the next month, Company is sure that it is not experiencing stockout. Although, it is the worst possible case to happen in stochastic lead time, Company is not penalised by the expensive air transmission costs. Henceforth, the stochastic inventory problem and the optimal order quantity could be found by solving problem (3). The other parts of this model are explained in the next section. Particularly, parameters P[j].patInf[i] and P[j].empty, and variables x_j are explained in Section 4.2.

4 THE CONTAINER LOADING PROBLEM

To solve the loading problem, two different integer programming (IP) approaches are considered. The first approach has two phases. First, the optimal number of containers is determined by an IP problem. In the second phase, another IP problem is solved to maximise volume utilization of containers. The first approach is a naive zero-one IP model of the problem. In the second approach, different patterns of packing a container is enumerated, and based on those patterns, a single pure IP problem finds the optimal solution. The second approach has huge computational superiority to the first one.

4.1 The first approach: Pallet assignment

The aim of the first phase of this approach is to find the optimal number of containers to carry the demand for raw materials. Here we describe a formulation that models this in terms of assigning pallets into containers.

Parameters.

- n_1, n_2, n_3 and n_4 are the amount of order for four types of raw materials.
- $N = \lceil n_1/7 \rceil + \lceil n_2/7 \rceil + \lceil n_3/7 \rceil + \lceil n_4/7 \rceil$. This number serves as an upper bound for the unknown number of containers. The number 7 comes from the fact that it is possible to load a container with at least seven item of any pallet type.
- Each container has three rows ($Row = 3$) and at most nine pallets can be placed in a container.
- m_1, m_2, m_3 and m_4 are loaded pallet weights, and M is the weight capacity of each container.

Variables.

- x_1^{ijk} are binary variables $\forall i(i = 1, \ldots, n_1), \forall j(j = 1, \ldots, N), \forall k(k = 1, \ldots, Row)$; They are 1 if item (pallet) i of the first raw material is placed in container j on row k. Otherwise they are 0. The binary variables x_2^{ijk}, x_3^{ijk} and x_4^{ijk} are defined in the same manner.

Constraints of the first problem.

1. All of the demand should be packed in the containers: $\sum_{j=1}^{N} \sum_{k=1}^{Row} \sum_{i=1}^{n_s} x_s^{ijk} = n_s, \quad s = 1, 2, 3, 4.$
2. Each item should be packed once: $\sum_{j=1}^{N} \sum_{k=1}^{Row} x_s^{ijk} = 1 \quad \forall i = 1, \ldots, n_s, \quad s = 1, 2, 3, 4.$
3. There should be at most nine pallets in a container:
$$\sum_{k=1}^{Row} [\sum_{i=1}^{n_1} x_1^{ijk} + \sum_{i=1}^{n_2} x_2^{ijk} + \sum_{i=1}^{n_3} x_3^{ijk} + \sum_{i=1}^{n_4} x_4^{ijk}] \leq 9 \quad \forall j.$$
4. There are three rows in a container: $\sum_{i=1}^{n_1} x_1^{ijk} + \sum_{i=1}^{n_2} x_2^{ijk} + \sum_{i=1}^{n_3} x_3^{ijk} + \sum_{i=1}^{n_4} x_4^{ijk} \leq 3 \quad \forall j, k$
5. Weight capacity of containers should be respected:
$$\sum_{k=1}^{Row} \left[m_1 \sum_{i=1}^{n_1} x_1^{ijk} + m_2 \sum_{i=1}^{n_2} x_2^{ijk} + m_3 \sum_{i=1}^{n_3} x_3^{ijk} + m_4 \sum_{i=1}^{n_4} x_4^{ijk} \right] \leq M \quad \forall j.$$
6. The pallets have two categories that pallets of the same category can be stacked on top of each other but different categories cannot: $x_s^{ijk} + x_t^{ijk} \leq 1 \quad \forall i, k, j,$ and $(s, t) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.

The objective functions. For the first stage we try to minimize the empty spaces in all the containers to find the optimal number of containers. Specially, as the maximum number of slots in a container is nine, we are interested in minimising vacant slots in each container, summed up over all of them. Therefore, the objective function of the first phase is

$$\min \sum_{j=1}^{N} \left(9 - \sum_{k=1}^{Row} [\sum_{i=1}^{n_1} x_1^{ijk} + \sum_{i=1}^{n_2} x_2^{ijk} + \sum_{i=1}^{n_3} x_3^{ijk} + \sum_{i=1}^{n_4} x_4^{ijk}] \right).$$

The second phase has almost the same structure with some minor differences. The optimal number of containers is fixed, and Company can provide their priorities for different raw materials to fill the empty volume using the most valuable material. The objective function of this IP problem is

$$\max \sum_{j=1}^{N} \sum_{k=1}^{Row} \left(\sum_{i=1}^{n_1} x_1^{ijk} + \sum_{i=1}^{n_2} x_2^{ijk} + \sum_{i=1}^{n_3} x_3^{ijk} + \sum_{i=1}^{n_4} x_4^{ijk} \right)$$

at the moment, all the materials have the same priorities.

4.2 The second approach

The second approach to formulate and solve the loading problem is very interesting from computational point view. One reason for the computational superiority is that the problem has a nice symmetry in loading pallets into containers. Avoiding this property complicates the model by introducing huge number of variables and constraints which are highly dependent on the order size. However, in the second model, the number of decision variables and constraints are independent from the order size for raw materials. Firstly, the number of all different patterns of loading a single container is enumerated. While this is in principle exponential, in practice it can be done in very little CPU time for the datasets encountered in this application. All the physical constraints are considered in the enumeration of the patterns. Then, a pure integer programming problem is modeled to find the optimal number of containers so that the volume utilization is maximized.

4.3 The second approach: Container patterns

The second approach to formulate and solve the loading problem is very interesting from computational point view. One reason for the computational superiority is that the problem has a large amount of symmetry in loading pallets into containers which significantly increases the size of the branch and bound tree for the first approach. In addition the model requires an initial guess of the number of containes required which can also significantly affect performance. In the second model, the number of decision variables and constraints are independent from the order size for raw materials. Firstly, the number of all different patterns of loading a single container is enumerated. While this is in principle exponential, in practice it can be done in very little CPU time for the datasets encountered in this application. All the physical constraints are considered in the enumeration of the patterns. Then, a pure integer programming problem is modeled to find the optimal number of containers so that the volume utilization is maximized.

Patterns. Patterns are defined as a class in a Python program. It counts all possible arrangements in a container considering the weight limitation, possibility of packing appropriate pallets types on each other, and that at most nine possible positions exist in a container. Each member of the class has several attributes. For example, P[i].weight represents the weight of pattern i, P[i].empty indicates number of empty positions out of nine in a container if pattern i is utilised, and P[i].patInf[j] shows the number of pallets of type $j \in \{1, 2, 3, 4\}$ in pattern i.

Parameters.

- n_1, n_2, n_3 and n_4 are the amount of order for four types of raw materials.
- $N = \lceil n_1/7 \rceil + \lceil n_2/7 \rceil + \lceil n_3/7 \rceil + \lceil n_4/7 \rceil$. This number serves as an upper bound for the unknown number of containers.
- P is the number of different packing patterns in a container.

Variables.

- x_i is an integer variable ;

Constraints of the first problem.

1. All of the demand should be packed in the containers:$\sum_{i=1}^{P} x_i$P[i].patinfo[j] $\geq n_j, \forall j \in \{1, 2, 3, 4\}$, where, the number of j^{th} raw material in the pattern P[i] is denoted by P[i].patinfo[j].

The objective function. In this formulation, all other constraints are considered in the calculation of different patterns for a single container. The objective function of this model consists of minimizing the number of containers and the number of empty spaces in the containers simultaneously by $\min \sum_{i=1}^{P} x_i +$ P[i].empty. This model is always have four constraints and 428 variables, independent from the order size.

4.4 Embedding the container packing problem in the stochastic optimisation model

The second formulation for the container packing problem is computationally promising. Regardless of the size of order quantity, it has same number of variables and constraints. This creates the opportunity to embed this formulation of container packing problem into the stochastic optimisation model for the inventory problem and construct model (3) and solve both problems at the same time. Therefore, the optimal order quantity for the demand, and the optimal container packing with maximum volume utilisation is calculated simultaneously.

5 Numerical results and Conclusion

In this paper, we developed a stochastic optimisation model to represent a probabilistic inventory problem with stochastic demand and lead time, combined with a container packing problem. In order to solve the container packing aspect of the problem, we developed two integer programming formulations. The models are then solved by MIP solver of Cplex package. The container pattern based model has less and constant number of variables and constraints independent of demand, unlike the pallet assignment formulation. For example, for a $n_1 = n_2 = n_3 = n_4 = 10$, the pallet assignment model has 9680 constraints and 960 binary variables in the reduced MIP for the first stage, and 52738 constraints and 3888 binary variables for the second stage. By contrast the container pattern model has only four constraints and 428 integer variables, and takes 0.2 seconds to find the optimal packing. The first model takes 97 seconds for this small problem. For the next step of research in this area, we have plan to solve the stochastic optimisation problem to find the appropriate order size for the uncertain inventory problem and the optimal packing simultaneously. By introducing different scenarios for the demand, there is a nice decomposable structure to utilise more efficient integer programming algorithms such as column generation. Also, it is worthwhile to mention that the work is being utilised by a manufacturing company to cope with several uncertainties in their production activities.

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