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Modelling the Relationship between Duration and Magnitude of Changes in Asset Prices

F. Chan [a], **J. Petchey**[a]

[a]*School of Economics and Finance, Curtin University, GPO BOX U1987, Perth, Western Australia, 6845*
Email: James.Petchey@curtin.edu.au

Abstract: This paper examines the relationship between the duration and magnitude of changes in asset prices using ultra-high frequency data. The literature on modelling conditional duration of changes in asset prices focuses mainly on the past durations without utilising any additional information. Similarly, the conditional models for changes in asset price do not take into account the duration of the changes. Given both variables contain information regarding market movement and investors' sentiment, it seems natural to test if past durations contains any information for the magnitude of price changes and if the magnitude of previous price changes contain any useful information in predicting the duration of the next price change.

This paper proposes a new model that captures the interaction between duration and magnitude of changes in asset prices, and thus provides a convenient framework to test statistically the existence of such relationship. The model is flexible and contains various well known models as special cases, including, the Exponential Generalised Autoregressive Heteroskedasticity (EGARCH) model of Nelson (1991) and the Logarithmic Conditional Duration (Log-ACD) model of Bauwens and Giot (2000). Despite having the EGARCH model as a special case, the objective of the model is not trying to model conditional duration and conditional volatility jointly. As shown in Ghysels and Jasiak (1998), modelling conditional duration and volatility jointly is technically challenging. This is due to the fact that volatility is defined over a regular sampling frequency but duration is defined over irregular time intervals. Given GARCH model is not generally closed under temporal aggregation, this creates a challenging modelling problem. The aim of this paper is to avoid this challenge by not modelling the conditional volatility, but instead, model the dynamics in the magnitudes of price change. The paper argues that since volatility is a function of the magnitudes of price change, testing the relationship between duration and the magnitude of price change provides an indirect test on the relationship between duration and volatility.

The paper also obtains theoretical results for the Quasi-Maximum Likelihood Estimator (QMLE) for the proposed model. Specifically, sufficient conditions for consistency and asymptotic normality are derived under mild assumptions. Monte Carlo experiments also provide further support of the theoretical results and demonstrate that the QMLE has reasonably good finite sample performance.

The paper then applies the model to nine different assets from three different asset classes, namely two exchange rate, two commodities and five stocks. The two currencies are Australia/US and British Pound/US exchange rates; the two commodities are Gold and Silver and the five stocks are BHP, Rio Tinto, CBS, ANZ and Apple. The sample spans from 4 January 2010 to 30 December 2011 with an average of 100,000 observations.

Keywords: *Intra-daily data, duration, volatility, price change*

1 INTRODUCTION

This paper examines the relationship between the duration and magnitude of changes in asset prices using ultra-high frequency data. Although there is a substantial literature on volatility modelling and duration between price change, there has been surprisingly little investigation on the relationship between the two. Interestingly, there are similarities in the specifications of modelling conditional volatility and conditional duration. Specifically, the Autoregressive Conditional Duration (ACD) model of Engle and Russell (1998) is similar, in terms of its dynamics specifications, to the popular Generalised Autoregressive Conditional Heteroskedasticity (GARCH) model of Bollerslev (1986) and the specification of log-Autoregressive Conditional Duration (log-ACD) model is similar to the Exponential GARCH (EGARCH) model of Nelson (1991). Although this similarity may appear to provide a convenience way to model both volatility and duration jointly using intra-daily data, this is unfortunately not the case. As demonstrated in Ghysels and Jasiak (1998), it is technically challenging to estimate GARCH-type and ACD-type models jointly, despite their similarities in specification. This is due to the fact that volatility is defined over a regular sample frequency and duration is defined over irregular time intervals. Using the result that most GARCH type models are not closed under temporal aggregation as shown in Drost and Nijman (1993), Ghysels and Jasiak (1998) demonstrated the difficulties in constructing a model to capture the interdependency between duration and volatility.

While Ghysels and Jasiak (1998) has successfully combined the two models together, the success comes with a cost in estimation. Specifically, the estimation of ACD-GARCH model is not straightforward and more importantly, the GARCH process in the ACD-GARCH model is strictly speaking, no longer a model for conditional volatility, but rather an approximation to the underlying dynamics of price changes. The contribution of this paper is to provide a more convenient framework to test the relationship between duration and magnitudes of changes in asset price. Since the difference in price contains information about the underlying volatility, testing the interdependence between duration and the magnitude of price change provides an indirect test between duration and volatility.

Apart from proposing a new model, this paper also establishes sufficient conditions for consistency and asymptotic normality of Quasi Maximum Likelihood Estimator (QMLE) for the proposed model. The results are supported by a series of Monte Carlo Experiments. The paper also applies the model to nine different assets from three different asset classes, namely, two currencies, two commodities and five stocks. The two currencies are Australia/US and British Pound/US exchange rates; the two commodities are Gold and Silver and the five stocks are BHP, Rio Tinto, CBS, ANZ and Apple. The sample spans from 4 January 2010 to 30 December 2011 with an average of 100,000 observations.

The paper is organised as follows: Section 2 introduces the model and its properties. It also contains a proposition that provides sufficient conditions for consistency and asymptotic normality of QMLE for the proposed model. This is followed by some Monte Carlo simulation in Section 3. Empirical results will be presented in Section 4 and Section 5 contains some concluding remarks.

2 MODEL

This section proposes a model that provides a convenient framework to examine the interdependence between duration and magnitude of changes in asset prices. The notation used in the his paper will be introduced as follows. Let r_i denotes the return calculated on the i^{th} trade since the first observation with $v_i = r_i^2$ and x_i denotes the duration between the i^{th} and $(i-1)^{th}$ trades. For any matrix $Y = \{Y_{ij}\}$, $\log Y = \{\log Y_{ij}\}$ for all i, j and $vec(A)$ denotes the stack operator that converts the $m \times n$ matrix A to a $mn \times 1$ vector by stacking the columns of A. $\mathbb{E}(x)$ denotes the expectation of x with respect to an appropriate measure that should be clear in the context of the discussion. $\mathbf{i}$ denotes a column vector of 1s and $|A|$ denotes the determinant of the matrix A. $Y \sim LN(\mu, \Sigma)$ denotes a multivariate log-normal distribution such that $\log(Y)$ has mean μ and variance-covariance matrix Σ. Note that the definition of $\log Y$ is different to the standard definition in the literature, which defines $\log Y$ by using its Taylor expansion.

Let $Y_i = (v_i, x_i)'$, consider

$$Y_i = \tilde{\Phi}_i \varepsilon_i \quad (1)$$

where $\varepsilon_i = (\eta_i, \xi_i)'$ is a sequence of iid random vector with joint density $f(\varepsilon_i)$ which has non-negative supports. Moreover, $\mathbb{E}(\varepsilon_i) = \mathbf{i}$ for all i and $\tilde{\Phi}_i = diag(P_i^*)$ such that

$$P_i^* = \begin{pmatrix} h_i^* \\ \phi_i^* \end{pmatrix}. \quad (2)$$

Consider two possible processes for the log transform of P_i^*

$$\log P_i^* = \tilde{\Gamma}_0 + \sum_{j=1}^{p} \tilde{\Gamma}_j \tilde{\varepsilon}_{i-j} + \sum_{j=1}^{q} \tilde{\Lambda}_k \log P_{i-k}^* \tag{3}$$

and

$$\log P_i^* = \tilde{\Gamma}_0 + \sum_{j=1}^{p} \Gamma_j \log Y_{i-j} + \sum_{k=1}^{q} \Pi_k \log P_{i-k}^* \tag{4}$$

where $\tilde{\varepsilon}_i = \left(\sqrt{\eta_i}, \sqrt{\xi_i}\right)'$ and $\tilde{\Gamma}_0$ is a 2×1 vector. Γ_j, $\tilde{\Gamma}_j$, $\tilde{\Pi}_k$ and Π_k are 2×2 coefficient matrices for $j = 1, ..., p$ and $k = 1, ..., q$. Equation (3) depends on the unobserved $(\eta_i, \xi_i)'$ whereas equation (4) depends on the observed data. Under the assumption that $\mathbb{E}(r_i) = 0$, v_i can be interpreted as volatility estimates sampled from irregular intervals. Theoretically, the equation that governs the behaviour of v_i in(3) can be interpreted as the symmetric version of the EGARCH process as proposed in Nelson (1991) under appropriate parameter restrictions. Similarly, the equation that governs the behaviour of duration in (4) is an extension to the log-ACD model as proposed in Bauwens and Giot (2000). To make this point clearer, consider equation (3) in the following form:

$$\begin{pmatrix} \log h_i^* \\ \log \phi_i^* \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{0,1} \\ \tilde{\gamma}_{0,2} \end{pmatrix} + \sum_{j=1}^{p} \begin{pmatrix} \tilde{\gamma}_{j,11} & \tilde{\gamma}_{j,12} \\ \tilde{\gamma}_{j,21} & \tilde{\gamma}_{j,22} \end{pmatrix} \begin{pmatrix} \sqrt{\eta}_{i-j} \\ \sqrt{\xi}_{j-1} \end{pmatrix} + \sum_{k=1}^{q} \begin{pmatrix} \tilde{\lambda}_{k,11} & \tilde{\lambda}_{k,12} \\ \tilde{\lambda}_{k,21} & \tilde{\lambda}_{k,22} \end{pmatrix} \begin{pmatrix} \log h_{i-k}^* \\ \log \phi_{i-k}^* \end{pmatrix}. \tag{5}$$

Note that $\sqrt{\eta}_i = |r_i|/\sqrt{h_i}$ and $\sqrt{\xi}_i = \sqrt{x}_i/\sqrt{\phi}_i$, which implies

$$\begin{pmatrix} \log h_i^* \\ \log \phi_i^* \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{0,1} \\ \tilde{\gamma}_{0,2} \end{pmatrix} + \sum_{j=1}^{p} \begin{pmatrix} \tilde{\gamma}_{j,11} & \tilde{\gamma}_{j,12} \\ \tilde{\gamma}_{j,21} & \tilde{\gamma}_{j,22} \end{pmatrix} \begin{pmatrix} |r_{i-j}|/\sqrt{h}_{i-j} \\ \sqrt{x}_{i-j}/\sqrt{\phi}_{i-j} \end{pmatrix} + \sum_{k=1}^{q} \begin{pmatrix} \tilde{\lambda}_{k,11} & \tilde{\lambda}_{k,12} \\ \tilde{\lambda}_{k,21} & \tilde{\lambda}_{k,22} \end{pmatrix} \begin{pmatrix} \log h_{i-k}^* \\ \log \phi_{i-k}^* \end{pmatrix}. \tag{6}$$

Set $\tilde{\gamma}_{j,12} = \tilde{\lambda}_{k,12} = 0$ for all j, k then $\log v_i$ follows the symmetric version of the EGARCH process. Similarly, equation (4) can be expressed as:

$$\begin{pmatrix} \log h_i^* \\ \log \phi_i^* \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{0,1} \\ \tilde{\gamma}_{0,1} \end{pmatrix} + \sum_{j=1}^{p} \begin{pmatrix} \gamma_{j,11} & \gamma_{j,12} \\ \gamma_{j,21} & \gamma_{j,22} \end{pmatrix} \begin{pmatrix} \log v_{i-j} \\ \log x_{i-j} \end{pmatrix} + \sum_{k=1}^{q} \begin{pmatrix} \pi_{k,11} & \pi_{k,12} \\ \pi_{k,21} & \pi_{k,22} \end{pmatrix} \begin{pmatrix} \log h_{i-k}^* \\ \log \phi_{i-k}^* \end{pmatrix} \tag{7}$$

and by setting $\gamma_{j,12} = \pi_{k,12} = 0$ for all j and k, $\log \phi_i$ follows the log-ACD model as defined in Bauwens and Giot (2000). However, the un-observability of ξ_i and η_i in (3) makes it difficult to derive the appropriate structural and statistical properties and equation (4) is much easier to analyse theoretically. For this reason, this paper will be focusing on equation (4) only.

Note that equation (4) can be rewritten differently to avoid potential identification problems. Since $\log Y_i = \log P_i^* + \log \varepsilon_i$, therefore equation (4) can be rewritten as:

$$\log P_i^* = \tilde{\Gamma}_0 + \sum_{j=1}^{p} \Gamma_j \log \varepsilon_{i-j} + \sum_{k=1}^{\max(p,q)} \Lambda_k \log P_{i-k}^* \tag{8}$$

where $\Lambda_k = \Pi_k + \Gamma_k$ for $k \le p$ with Π_k and Γ_j being the null matrix for $k > q$ and $j > p$, respectively.

The assumption that $\mathbb{E}(\varepsilon_i) = \mathbf{i}$ can be relaxed. To see this, assume that $\mathbb{E}(\varepsilon_i) = \varepsilon = (\eta, \xi)'$ such that $\eta, \xi > 0$, then

$$Y_i = \tilde{\Phi}_i \varepsilon_i \equiv \Phi_i \nu_i \tag{9}$$

where $\nu_i = (\eta_i/\eta, \xi_i/\xi)'$ and $\Phi_i = diag(P_i)$ with

$$P_i = \begin{pmatrix} h_i \\ \phi_i \end{pmatrix} \tag{10}$$

and

$$\log P_i = \Gamma_0 + \sum_{j=1}^{p} \Gamma_j \log Y_{i-j} + \sum_{k=1}^{q} \Pi_k \log P_{i-k} \tag{11}$$

such that $\Gamma_0 = \tilde{\Gamma}_0 + (\mathbf{I} - \sum_{k=1}^{q} \Pi_k) \log \varepsilon$. Although this is structurally valid, the estimation of Γ_0 may be problematic as it suffers from identification issues. However, the accurate estimate of Γ_0 is often not important in practice as the focus tends to be on the consistent estimation of the coefficient matrices, namely, Γ_j and Π_k, and their associated inferences.

In terms of estimation, this paper proposes to estimate the parameters of the model as defined by equations (1) and (4) using (Quasi) Maximum Likelihood Estimator ((Q)MLE). This requires an assumption on the joint density of ε_i and this paper assumes $\varepsilon_i \sim LN(\mathbf{0}, \Sigma)$. There are two reasons for choosing the multivariate log-normal distribution. Firstly, there exists statistical properties for the log-ACD model. Specifically, the consistency and asymptotic normality of QMLE have been obtained for the log-ACD model under the log-normal distribution as shown in Allen et al. (2008). Given the close relationship between the proposed model and the log-ACD as demonstrated in previous section, this paper is able to obtain consistency and asymptotic normality of QMLE for the proposed model by extending the arguments adopted in Allen et al. (2008). Secondly, the multivariate log normal distribution is relatively simple but flexible enough to provide a good approximation of other distributions. This is quite important as the true distribution of both duration and changes in price are unknown, so a flexible distribution that can provide adequate approximations to other distributions is desirable. More importantly, consistency and asymptotic normality of QMLE in this case will facilitate valid inference greatly, even thought the underlying joint distribution may be different to the multivariate log-normal distribution.

Define $\theta = (vec(\Gamma)', vec(\Lambda)', vec(\Sigma))'$. This implies the *conditional* probability density is

$$f(\varepsilon_i) = (2\pi)^{-1} |\Sigma|^{-1} (\eta_i \xi_i) \exp\left[-\frac{1}{2} \log \varepsilon_i' \Sigma^{-1} \log \varepsilon_i \right] \tag{12}$$

where $\varepsilon_i = (\eta_i, \xi_i)' = (v_i/h_i, x_i/\phi_i)'$ and Σ is the variance covariance matrix between $\log \eta_i$ and $\log \xi_i$. Thus the log-likelihood function given n observations is

$$l(\theta, Y) = -n \log 2\pi - n \log |\Sigma| - \sum_{i=1}^{n} (\log \eta_i(\theta) + \log \xi_i(\theta)) - \frac{1}{2} \sum_{i=1}^{n} \left[\log \varepsilon_i(\theta)' \Sigma^{-1} \log \varepsilon_i(\theta) \right] \tag{13}$$

and hence the (Q)MLE of θ, $\hat{\theta}$ is

$$\hat{\theta} = \arg\max_{\theta} l(\theta, Y). \tag{14}$$

The following assumptions are sufficient to establish the statistical properties of Model as specified in equations (1) and (3).

Assumption 1. Θ is an open and compact subset of the Euclidean space $\mathbb{R}^{2+4(p+q)}$.

Assumption 2. Y_i is stationary and ergodic.

Assumption 3. $\Gamma(L)$ and $\Pi(L)$ are left co-prime where $\Gamma(L) = \sum_{j=1}^{p} \Gamma_j L^j$ and $\Pi(L) = \sum_{j=1}^{q} \Pi_j L^j$.

Assumption 4. $\Gamma(1) + \Pi(1) < 1$.

Assumption 5. $\mathbb{E} \log \varepsilon_i = \mathbf{0}$.

Assumption 6. $0 < |\Sigma| < \infty$.

Assumptions (1) - (6) are standard in the literature. Conditions for Assumption (2) would be valuable but it is beyond the scope of this paper.

Table 1. Descriptive Statistics on the Parameter Estimates from the Monte Carlo Experiments

Variables	Mean	Deviation	Skewness	Kurtosis	JB Test	JB P-value
$\gamma_{1,11}$	0.294	0.010	0.001	2.952	0.098	0.952
$\gamma_{1,21}$	-0.508	0.010	0.001	3.186	1.446	0.485
$\gamma_{1,12}$	0.185	0.009	-0.001	2.899	0.428	0.807
$\gamma_{1,22}$	-0.405	0.009	-0.001	2.888	0.526	0.769
$\pi_{1,11}$	0.644	0.019	-0.000	3.475	9.408	0.009
$\pi_{1,21}$	0.264	0.017	-0.003	3.598	14.907	0.001
$\pi_{1,12}$	-0.393	0.014	-0.004	3.193	1.560	0.458
$\pi_{1,22}$	0.433	0.027	-0.003	3.297	3.678	0.159
σ_{11}	1.265	0.037	0.006	3.510	10.840	0.004
σ_{21}	0.761	0.033	0.005	3.038	0.064	0.968
σ_{12}	0.761	0.033	0.005	3.038	0.064	0.968
σ_{22}	1.266	0.037	0.004	2.988	0.008	0.996

Proposition 1. *Under the Assumptions (1) - (4), let θ_0 be the true values of the parameters in the process Y_i as defined in equations (1) and (3), then the Quasi-Maximum Likelihood Estimator (QMLE) as defined in equation (14) is consistent, that is, $\hat{\theta} \xrightarrow{p} \theta_0$. Moreover, $\sqrt{n}\left(\hat{\theta} - \theta_0\right) \sim N\left(0, \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right)$ where*

$$\mathbf{A} = \left.\frac{\partial^2 l}{\partial\theta\partial\theta'}\right|_{\theta=\hat{\theta}} \qquad \mathbf{B} = \left.\frac{\partial l}{\partial\theta}\frac{\partial l}{\partial\theta'}\right|_{\theta=\hat{\theta}}.$$

3 MONTE CARLO EVIDENCE

This section provides some Monte Carlo evidence supporting the theoretical results as presented in Proposition (1). The experiment examines the case when $p = q = 1$ with the number of observations equals to 5000 with 1000 replications. The coefficient matrices used in the experiments can be found as follows:

$$\Gamma_1 = \begin{pmatrix} 0.3 & -0.5 \\ 0.2 & -0.4 \end{pmatrix} \qquad \Lambda_1 = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.5 \end{pmatrix} \qquad \Gamma_0 = \begin{pmatrix} 0.4 \\ 0.1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Table 1 summarised the results of the experiments. As shown in Table 1, most parameter estimates are very close to their truth values. The Jarque-Bera test cannot reject the null of normality for all parameter estimates except for $\pi_{1,11}$, $\pi_{1,21}$ and σ_{11}. These are mostly due to excess kurtosis in the distribution of the parameter estimates and will decrease as the number of observations increases.

4 EMPIRICAL RESULTS

This section examines the relationship between duration and changes in asset prices using the model presented in Section 2. The sample period spans between 4 January 2010 and 30 December 2011. The data includes all trades between 10:30am and 4:00pm for each day within the sample period. The paper considers nine assets from three different asset classes, namely two currencies, two commodities and five stocks. The two currencies are Australia/US and British Pound/US exchange rates; Gold and Silver are the two commodities and the five stocks are BHP, Rio Tinto, CBS, ANZ and Apple. The average number of observations are 100,000.

Tables (2) and (3) contain the parameter estimates for the nine assets. As shown in the tables, the statistical significance of α_{21} estimates suggests that price changes generally have a negative impact on duration in the short run, with the exception of Gold. This implies large changes in price leads to shorter duration in the short run. Its long run impact, however, is less obvious and somewhat mixed as demonstrated by the statistical significance of β_{21} estimates. Only three out of the nine assets have statistically significance β_{21} estimates, namely, BHP, Gold and Silver. A possible explanation of this mixed results might be due to possible asymmetric response from duration to price changes. That is duration may response differently between a positive and negative price change.

Interestingly, there is evidence that duration can affect the magnitude of price change in the short run as suggested by the statistical significant of α_{12} estimates. However, it is not obvious if this impact is positive or negative overall. Four out of nine assets have positive estimates for α_{12}, namely AUD/US, GBP/US, Gold and

ANZ. This suggests that long duration leads to larger price change for these four assets, whereas long duration leads to smaller price change for Silver, BHP, Rio Tinto, CBA and Apple.

The results of the Σ estimates are also mixed. Interestingly, the σ_{12} estimates are statistically significant for AUD, Gold and Rio Tinto and Apple but not significant for BHP, CBA, ANZ, GBP and Silver. The signs of the covariance are also mixed. These results suggest that the relationship between duration and the magnitude of price change is highly asset dependent and there does not seem to be any stylised fact which can be derived from these results.

5 CONCLUSION

This paper proposed a new model for duration and magnitude of changes in asset prices. The model utilised ultra-high frequency data and provided a convenient framework to examine the relationship between duration and the magnitude of price changes. Theoretical results showed that QMLE is consistent and asymptotically normal for the parameters of the proposed model and these results were supported by Monte Carlo experiments. Compare to other existing models, the proposed model is simpler in terms of estimation and hypothesis testing. Empirical results show that the magnitude of price changes were correlated with future duration and vice versa. This will have significant implications in establishing forecast model for duration and intra-daily volatility.

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Table 2. Parameter Estimates for Currencies and Commodities

	Parameter	AUD	GBP	Gold	Silver
Γ_1	$\gamma_{1,11}$	0.263	0.24	0.0714	0.143
		(63.2)	(31.6)	(28.0)	(4.64)
	$\gamma_{1,21}$	-0.102	-0.0217	0.00438	-0.0724
		(-37.2)	(-11.1)	(6.19)	(-3.6)
	$\gamma_{1,12}$	0.0114	0.0341	0.0514	-0.0628
		(4.25)	(11.3)	(19.4)	(-3.49)
	$\gamma_{1,22}$	-0.00561	-0.0117	0.0153	0.0408
		(-4.83)	(-5.81)	(22.8)	(3.42)
Π_1	$\pi_{1,11}$	0.954	0.756	0.899	-0.471
		(16.0)	(86.8)	(226.0)	(-6.33)
	$\pi_{1,21}$	-0.0109	0.0075	-0.00857	0.0698
		(-0.609)	(1.71)	(-10.4)	(2.77)
	$\pi_{1,12}$	3.31	1.83	-0.249	-2.2
		(37.3)	(44.6)	(-10.4)	(-16.6)
	$\pi_{1,22}$	-0.492	-0.692	0.952	0.99
		(-12.7)	(-17.4)	(550.0)	(81.1)
Γ_0	$\gamma_{0,1}$	-0.0599	-0.076	0.226	-0.633
		(-0.132)	(-3.56)	(15.2)	(-5.7)
	$\gamma_{0,2}$	-0.101	0.0183	0.0415	-0.118
		(-0.648)	(1.19)	(16.6)	(-2.5)
Σ	σ_{11}	1.62	2.14	2.47	1.36
		(230.0)	(205.0)	(142.0)	(27.7)
	σ_{21}	-0.0567	-0.00132	0.064	-0.0455
		(-9.87)	(-0.167)	(13.2)	(-1.77)
	σ_{12}	-0.0567	-0.00132	0.064	-0.0455
		(-9.87)	(-0.167)	(13.2)	(-1.77)
	σ_{22}	1.53	1.82	0.915	1.46
		(120.0)	(129.0)	(106.0)	(62.3)
Log-likelihood		-454,136.655	-503,922.383	-452,191.3532	-426,820.7550

*t-statistics are in parenthesis

Table 3. Parameter Estimates for Stocks

	Parameter	BHP	Rio Tinto	CBA	ANZ	Apple
Γ_1	$\gamma_{1,11}$	0.435	0.265	0.277	0.338	0.0348
		(22.2)	(17.6)	(21.2)	(22.0)	(20.0)
	$\gamma_{1,21}$	-0.0399	-0.201	-0.201	-0.0477	-0.0412
		(-1.11)	(-20.8)	(-14.2)	(-2.06)	(-11.0)
	$\gamma_{1,12}$	-0.508	-0.0522	-0.0586	0.0139	-0.0225
		(-52.8)	(-14.3)	(-36.1)	(7.73)	(-31.2)
	$\gamma_{1,22}$	0.0525	0.0394	0.133	0.0106	0.23
		(2.73)	(7.34)	(19.1)	(2.42)	(44.2)
Π_1	$\pi_{1,11}$	0.735	1.15	0.636	0.12	0.961
		(411.0)	(4.17)	(32.8)	(1.18)	(319.0)
	$\pi_{1,21}$	0.655	-0.173	-0.045	0.0386	-0.0127
		(53.2)	(-0.718)	(-0.387)	(0.591)	(0.0414)
	$\pi_{1,12}$	0.132	1.14	0.14	-0.381	-0.717
		(14.0)	(2.44)	(11.0)	(-3.51)	(24.3)
	$\pi_{1,22}$	-0.0734	-0.00161	0.332	0.975	0.549
		(-1.1)	(-0.004)	(2.5)	(29.4)	(30.7)
Γ_0	$\gamma_{0,1}$	-0.242	0.0924	-0.865	-2.16	-0.0475
		(-1.79)	(0.11)	(-13.9)	(-7.86)	(-3.04)
	$\gamma_{0,2}$	-0.254	-0.155	-0.295	-0.0001	-0.251
		(-0.303)	(-0.204)	(-0.655)	(-0.0004)	(-1.65)
Σ	σ_{11}	0.492	1.19	0.819	0.277	2.06
		(365.0)	(91.3)	(92.3)	(46.5)	(173.0)
	σ_{21}	-0.0166	0.0627	-0.0186	-0.00854	-0.0547
		(-1.49)	(3.81)	(-1.9)	(-1.76)	(-3.46)
	σ_{12}	-0.0166	0.0627	-0.0186	-0.00854	-0.0547
		(-1.49)	(3.81)	(-1.9)	(-1.76)	(-3.46)
	σ_{22}	4.61	6.64	5.43	5.36	4.29
		(45.6)	(49.8)	(88.2)	(53.7)	(158.0)
Log-likelihood		-509,951.7658	-597,146.1998	-544,880.9624	-572,914.9425	-610,705.904

*t-statistics are in parenthesis