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A BINOMIAL MODEL APPROACH TO MODELLING PORTFOLIO VOLATILITY IN CONTINUOUS TIME

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ABSTRACT

The present article offers a binomial model replication of Merton's (1969, 1973) model of portfolio selection allowing volatility in continuous time. Interestingly, the investor risk premium is modelled as a consequence of the mathematics of risk itself (rather than that of investors setting prices at the start of each investment period). The model reveals the inherent circularity of the CAPM as an explanation of investor volatility-return preferences. In the model, the outcome return on a risky asset is identified as having an idiosyncratic risk component. The model thereby challenges the traditional interpretation of Markowitz portfolio theory, namely that idiosyncratic risk variations cancel.

1 INTRODUCTION.

The present paper introduces a binomial representation of Merton's (1969, 1973) model for portfolio selection under uncertainty allowing continuous adjustment. The application of continuous time models to portfolio analysis by practitioners currently remains restricted – certainly when compared with the broad acceptance and application in practice (at least until more recently) of the traditional CAPM. In part, this may be due to the inherent mathematical nature of the continuous time analysis with a consequently restricted intuitive conviction for its directives. The present paper may therefore have a role to play in advancing the application of models that allow continuous adjustment to investors' portfolio selections – at least as complimentary to those that seek to model portfolio selection over a "single investment period".

The model illuminates the interface of asset returns and portfolio volatility. For example, the nature of the equity risk premium. Thus a log-wealth utility investor is characterized as selecting across assets so as to maximize the portfolio's geometric continuously-compounded growth rate. The selected portfolio is thereby a function of the individual asset variances and covariances, but is independent of the portfolio's overall volatility. In this sense, log-wealth utility is characterized as "risk neutral". Or, we can say, for log-wealth utility, the outcome risk creates its own reward.

A novelty of the model is that the reward for risk (volatility) is recognized as the result of the mathematics of risk itself, rather than as the result of investors setting prices at the commencement of a one-period model (in accordance with required risk-return tradeoffs). The outcome is that increases in idiosyncratic risk should lead to observed increases in sampled periodic returns. This prediction appears to be consistent with Malkiel and Xu's (1997) empirical findings.

The remainder of the article is arranged as follows. In the following section, we present our exponential growth model for volatile markets; while the section thereafter presents the implications of a log-wealth utility for investors' propensity to invest in such markets. The final section summarises the article.

2 THE EXPONENTIAL (CONTINUOUSLY COMPOUNDED) GROWTH MODEL

When the growth rate applied to a valuation is both normally distributed and applied continuously, the outcome valuation at the end of a time period may be represented as the starting valuation multiplied by *exp*(y) where y is normally distributed; or, stated alternatively, the starting valuation multiplied by $exp(\mu + x)$ where μ represents the underlying mean or "drift" exponential (continuously compounded) growth rate for the period and x is normally distributed about zero with standard deviation (volatility), σ (see, for example, Dempsey, 2002).[1]

Such a distribution of growth outcomes $exp(\mu + x)$ over a time period may alternatively be represented as the outcome of a binomial process, repeated over a sufficiently large number N of time sub-periods. In a binomial process, \$1 either grows to $exp(\mu/N + \sigma/\sqrt{N})$ or declines to

$exp(\mu/N - \sigma/\sqrt{N})$ with equal likelihoods over each time sub-period.[2] The outcome of investing $1 in a portfolio comprising equity stocks and bonds (with normally distributed exponential growth rates) in combination with a risk-free asset, over a certain time period may therefore be represented as the outcome of applying the probability-weighted operator in Figure 1 over a sufficiently large number of sub-time periods.

In Figure 1, μ_S, μ_B, σ_S and σ_B, represent, respectively, the mean continuously compounded growth rates and standard deviations about such rates for stocks and bonds; C_{SB} represents the correlation coefficient between the exponential growth rates for stocks and bonds; and ω_S, ω_B represent, respectively, the proportions of the dollar invested in stocks and bonds, and hence $(1- \omega_S - \omega_B)$. represents the proportion of the dollar invested in the risk-free asset; and r_f represents the risk-free continuously compounded growth rate. The term ¼ $(1 + C_{SB})$ in Figure 1 represents the probability of an up-movement of both stocks and bonds in conjunction, or a down-movement of both stocks and bonds in conjunction; and ¼ $(1 - C_{SB})$ represents the probability of an up-movement of either stocks or bonds in conjunction with a down-movement of the other asset.

In order to investigate the outcome of applying the double-binomial operator of Figure 1 over successive time periods, we first identify the operator as the double-binomial operator in the second column of Figure 2. On solving between the two operator expressions, we have on a straightforward inspection, the relationships[3]:

$$\mu_0 = \tfrac{1}{2}\left\{\begin{array}{l}\ln\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} \\ + \\ \ln\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\}\end{array}\right\} \quad (1)$$

$$\mu_1 = \tfrac{1}{2}\left\{\begin{array}{l}\ln\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} \\ + \\ \ln\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\}\end{array}\right\} \quad (2)$$

$$\sigma_0 = \tfrac{1}{2}\left\{\begin{array}{l}\ln\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} \\ - \\ \ln\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\}\end{array}\right\} \quad (3)$$

$$\sigma_1 = \tfrac{1}{2}\left\{\begin{array}{l}\ln\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} \\ - \\ \ln\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\}\end{array}\right\} \quad (4)$$

At the end of a second period, the range of possible wealth outcomes is depicted as in the third column of Figure 2 (the combination of probability-weighted growth operators leading to a particular wealth outcome is indicated in curly brackets).[4]

The double-binomial framework of Figure 2 represents our model for growth for a portfolio of risky stocks and risky bonds with a risk-free asset. In the following section, we turn to consider the propensity of investors to invest in such growth.

3 PORTFOLIO SELECTION BY EXPONENTIAL GROWTH SEEKING INVESTORS

Osborne (1964) in his introduction of Brownian motion as a model of share price behaviour argues that equal *intervals* of subjective sensation (such as pitch, brightness or noise) typically correspond to equal *ratios* of physical stimulus (for example, of sound frequency, or of light or sound intensity). Consistently, he considers that when investors consider changes to their wealth, they remain concerned with *percentage* changes to the reference point or base of their current or upgraded wealth status. The exponential (continuously compounded) growth function $exp(r\mathrm{T})$ captures the impact of applying a percentage change r continuously with certainty over a time period T to the upgraded wealth that is a consequence of the percentage change growth process itself.[5] Consistently, we shall in the central part of this article take it that, when faced with a range of growth rate possibilities for their current wealth $$W_0$, investors value each possibility in direct proportion to its continuously compounded growth rate. That is:

$$U(r\mathrm{T}) = r\mathrm{T} \quad (5)$$

where $U(rT)$ represents the *utility* afforded to the investor by the continuously compounded growth rate r applied with certainty over the time period T. Equation 5 is equivalent to the statement that investors are subject to a log-wealth utility function (which follows allowing that when an investor's current wealth W_0 is subject to a continuously compounded growth rate r over a time period T, the outcome wealth is $W_0\, exp(rT)$, so that the investor's change in log-wealth utility is determined as $\ln[W_0\, exp(rT)] - \ln(W_0) = rT$, consistent with the utility derived from equation 5).

The Von Neumann and Morgenstern (1947) theorem states that investors seek to maximise portfolio expected utility U_P as:

$$U_P = \sum_{i=1}^{N} p(x_i)\, u(x_i) \qquad (6)$$

where $p(x_i)$ is the probability of each possible wealth outcome x_i, and $u(x_i)$ is the corresponding utility. Combining equations 5 and 6, a log-wealth utility investor will seek to maximise expected utility (U_P) at the end of a single time period in column 2 of Figure 2 as: [6]

$U_P = \frac{1}{2}[(1+C_{SB}).\mu_0 + (1-C_{SB}).\mu_1]$ which may be expressed:

$$U_P = \mu_P \qquad (7)$$

where: $\mu_P = \frac{1}{2}[(1+C_{SB}).\mu_0 + (1-C_{SB}).\mu_1]$ (8)

represents the mean or probability-weighted continuously compounded growth rate over μ_0 and μ_1.

From column 3 of Figure 2, the probability-weighted continuously compounded growth rate, and thereby the expected utility of outcomes, over two successive periods is observed as: $\frac{1}{2}(1+C_{SB})^2.\mu_0 + \frac{1}{2}(1-C_{SB})^2.\mu_1 + \frac{1}{2}(1+C_{SB})(1-C_{SB}).(\mu_0+\mu_1) = (1+C_{SB}).\mu_0 + (1-C_{SB}).\mu_1 =$ (with equation 8) $2\mu_P$. Similarly, over N successive periods, the probability-weighted continuously compounded growth rate, and thereby the expected utility of outcomes, U_P is expressed:

$$U_P = N\mu_P \qquad (9)$$

Thus the investor's problem of maximizing the expected utility offered by an investment in stocks, bonds and the risk-free asset, reduces to that of maximizing the utility expression 9 with respect to the proportions ω_S, ω_B, $(1-\omega_S-\omega_B)$, respectively, held in these assets. We therefore proceed to transform the utility expression 9 in terms of ω_S and ω_B. To this end, we substitute back for the definitions of μ_0 and μ_1 (equations 1 and 2) in equation 8. The utility (Up) offered by a portfolio combination of equity stocks, bonds and a risk-free asset over N periods may therefore be represented as:

$$\begin{aligned} U_P = N * \tfrac{1}{4} [\, & (1+C_{SB}).\ln\{\omega_S.exp(\mu_S+\sigma_S) + \omega_B.exp(\mu_B+\sigma_B) + (1-\omega_S-\omega_B).exp(r_f)\} \\ & + (1-C_{SB}).\ln\{\omega_S.exp(\mu_S+\sigma_S) + \omega_B.exp(\mu_B-\sigma_B) + (1-\omega_S-\omega_B).exp(r_f)\} \\ & + (1-C_{SB}).\ln\{\omega_S.exp(\mu_S-\sigma_S) + \omega_B.exp(\mu_B+\sigma_B) + (1-\omega_S-\omega_S).exp(r_f)\} \\ & + (1+C_{SB}).\ln\{\omega_S.exp(\mu_S-\sigma_S) + \omega_B.exp(\mu_B-\sigma_B) + (1-\omega_S-\omega_B).exp(r_f)\}\,] \end{aligned} \qquad (10)$$

Allowing $exp(y)$ converges to $1 + y + \frac{1}{2}y^2$ with a sufficiently small y, while restricting the analysis to terms of order μ_S, μ_B, r_f, σ_S^2, σ_B^2, equation 10 leads with sufficiently short time increments (large N) to: [7]

$$\begin{aligned} U_P = N * \tfrac{1}{4} [\, & (1+C_{SB}).\ln\{\omega_S.(\mu_S+\sigma_S+\tfrac{1}{2}\sigma_S^2-r_f) + \omega_B.(\mu_B+\sigma_B+\tfrac{1}{2}\sigma_B^2-r_f) + 1 + r_f\} \\ & + (1-C_{SB}).\ln\{\omega_S.(\mu_S+\sigma_S+\tfrac{1}{2}\sigma_S^2-r_f) + \omega_B.(\mu_B-\sigma_B+\tfrac{1}{2}\sigma_B^2-r_f) + 1 + r_f\} \\ & + (1-C_{SB}).\ln\{\omega_S.(\mu_S-\sigma_S+\tfrac{1}{2}\sigma_S^2-r_f) + \omega_B.(\mu_B+\sigma_B+\tfrac{1}{2}\sigma_B^2-r_f) + 1 + r_f\} \\ & + (1+C_{SB}).\ln\{\omega_S.(\mu_S-\sigma_S+\tfrac{1}{2}\sigma_S^2-r_f) + \omega_B.(\mu_B-\sigma_B+\tfrac{1}{2}\sigma_B^2-r_f) + 1 + r_f\}] \end{aligned}$$

$$\begin{aligned} = N * \tfrac{1}{4} [\, & (1+C_{SB}).\ln\{1+2\omega_S.(\mu_S+\tfrac{1}{2}\sigma_S^2-r_f) + 2\omega_B.(\mu_B+\tfrac{1}{2}\sigma_B^2-r_f) + 2r_f \\ & \qquad - \omega_S^2.\sigma_S^2 - \omega_B^2.\sigma_B^2 - 2\omega_S.\omega_B.\sigma_S.\sigma_B\} \\ & + (1-C_{SB}).\ln\{1+2\omega_S.(\mu_S+\tfrac{1}{2}\sigma_S^2-r_f) + 2\omega_B.(\mu_B+\tfrac{1}{2}\sigma_B^2-r_f) + 2r_f \\ & \qquad - \omega_S^2.\sigma_S^2 - \omega_B^2.\sigma_B^2 + 2\omega_S.\omega_B.\sigma_S.\sigma_B\}] \end{aligned}$$

which (allowing $\ln(1 + y)$ converges to y with sufficiently small y) may be expressed:

$$\begin{aligned} U_P = N * [\, & \omega_S.(\mu_S+\tfrac{1}{2}\sigma_S^2) + \omega_B.(\mu_B+\tfrac{1}{2}\sigma_B^2) + (1-\omega_S-\omega_B).r_f \\ & - \tfrac{1}{2}\omega_S^2.\sigma_S^2 - \tfrac{1}{2}\omega_B^2.\sigma_B^2 - C_{SB}.\omega_S.\omega_B.\sigma_S.\sigma_B] \end{aligned} \qquad (11)$$

or as $U_P = N\mu_P$ (equation 9) where:

$$\mu_P = \omega_S.(\mu_S+\tfrac{1}{2}\sigma_S^2) + \omega_B.(\mu_B+\tfrac{1}{2}\sigma_B^2) + (1-\omega_S-\omega_B).r_f$$

$$- \tfrac{1}{2}\, \omega_S^2.\sigma_S^2 - \tfrac{1}{2}\, \omega_B^2.\sigma_B^2 - C_{SB}.\, \omega_S.\omega_B.\, \sigma_S.\sigma_B \quad (12)$$

On identifying:

(i) the continuously compounded growth rate R_i that delivers asset i's expected (or average) wealth outcome as:

$$R_i = \mu_i + \tfrac{1}{2}\, \sigma_i^2 \quad (13)$$

(as Black and Scholes, 1973; cf, for example, Jacquier, Kane and Marcus, 2003),

(ii) the growth rate R_P that delivers the portfolio's expected wealth outcome as the weighted-average of the portfolio's asset growth rates, R_i:

$$R_P = \omega_S.\, R_S + \omega_B.R_B + (1- \omega_S - \omega_B).r_f \quad (14)$$

(with R_S, R_B defined by equation 13, i = S, B),

(iii) σ_P^2 as the outcome variance of the portfolio's asset growth rates:

$$\sigma_P^2 = \omega_S^2.\sigma_S^2 + \omega_B^2.\sigma_B^2 + 2\, C_{SB}.\omega_S.\omega_B.\sigma_S.\sigma_B, \quad (15)$$

the portfolio utility expression 11 is expressed:

$$U_P = N\,(R_P - \tfrac{1}{2}\, \sigma_P^2) \quad (16)$$

(which corresponds in continuous time with the expression for discrete time periods, for example, Cuthbertson, 1997, pg 55).

The terms between addition signs in equation 11 increase linearly with time (see footnote 2). It follows that the expression 11 is independent of the number of subperiods (N) over which the investment time horizon is divided. It follows similarly that utility increases linearly with the allocated time horizon – and hence that utility per period is independent of the investor's time horizon. The analysis thereby accords with Samuelson's arguments (1963, 1989, 1994), that a log-wealth utility investor's portfolio choice is *indifferent* to the investor's time-horizon.

The solution to maximizing the utility equation 11 with respect to ω_S and ω_B is identified by differentiating the expression with respect to both ω_S and ω_B and setting each outcome equation equal to zero. On differentiating the equation first with respect to ω_S, we obtain:

$$\mu_S + \tfrac{1}{2}\, \sigma_S^2 - r_f - \omega_S.\sigma_S^2 - C_{SB}.\omega_B.\, \sigma_S.\sigma_B = 0$$

which on identifying the covariance $\sigma_{S,B}$ between asset growth rates on stocks and bonds as:

$$\sigma_{S,B} = C_{SB}.\sigma_S\, .\sigma_B \quad (17)$$

may be expressed:

$$R_S - r_f - \omega_S.\, \sigma_S^2 - \omega_B.\, \sigma_{S,B} = 0 \quad (18a)$$

Similarly on differentiating equation 11 with respect to ω_B, we obtain:

$$R_B - r_f - \omega_S.\, \sigma_{B,S} - \omega_B.\, \sigma_B^2 = 0 \quad (18b)$$

With the covariance [$\sigma_{i,P}$ (i = S, B)] of the exponential growth rate for asset i with the exponential growth rate for the investor's portfolio P (comprising stocks, bonds and the risk-free assets in proportions, ω_S, ω_B, $1\text{-}\omega_S\text{-}\omega_B$) identified as:

$$\sigma_{i,P} = \omega_S.\, \sigma_{i,S} + \omega_B.\, \sigma_{i,B}$$

equations 18 are expressed:

$$R_S - r_f = \sigma_{S,P} \quad (19a)$$
$$R_B - r_f = \sigma_{B,P} \quad (19b)$$

On multiplying equation 19a by ω_S and equation 19b by ω_B, and adding the outcome equations, we arrive (with equation 14) at:

$$[R_P - r_f\,] = \sigma_P^2 \quad (20)$$

where σ_P^2 is the variance of the market portfolio. Alternatively, equations 18 for the optimal weights [ω_S, ω_B] may be expressed in matrix notation as:

$$W = \Omega^{-1}\, R \quad (21)$$

where:

W = the vector of weights [ω_S, ω_B] for the risky assets S and B,

Ω = the variance-covariance matrix:

$$\begin{bmatrix} \sigma_S^2 & \sigma_{S,B} \\ \sigma_{B,S} & \sigma_B^2 \end{bmatrix}$$

for the risky assets S and B, and

R = the vector of expected returns over the risk-free rate [R_S- r_f , R_B- r_f] for the risky assets S and B. (Equation 21 represents Merton's (1969) result, his equation 60.)[8]

For simplicity, the above analysis has been developed for the asset classes (S), bonds (B) and a risk-free asset. On subdividing the asset classes (S, B) into component assets, the analysis generalizes straightforwardly to a portfolio of N risky assets.

In accordance with equations 21, individual investors make portfolio allocations ω_i [i = S, B] (the dependent variables) dependent on their expectations for asset growth rates R_i [i = S, B] and their risk as identified by the covariance matrix of asset growth rates (the independent variables). The CAPM follows on a reconfiguration of the relationship between the dependent and independent variables in equations 21, combined with the assumption of homogenous expectations, for then all investors hold the market portfolio and equations 19 are expressed:

$$R_S = r_f + \sigma_{S,M} \quad (22a)$$
$$R_B = r_f + \sigma_{B,M} \quad (22b)$$

where $\sigma_{i,M}$ represents the covariance of the growth rate R_i (i = S, B) with the growth rate of the market portfolio R_M

(inclusive of the risk-free asset). Equation 20 then becomes:

$$[R_M - r_f] = \sigma_M^2 \qquad (23)$$

where σ_M^2 is the variance of the market portfolio. Thus with an asset's beta β_i defined as:

$$\beta_i = \sigma_{i,M} / \sigma_M^2 \qquad (24)$$

we may write equations 22 as:

$$R_i = r_f + \beta_i * [R_M - r_f] \quad (i = S, B) \qquad (25)$$

which is the traditional form of the CAPM applied to continuously compounded growth rates (as Merton, 1973).

The essential equations of this section relating to investor log-wealth utility and the outcome risk-return characteristics of portfolio composition may conveniently be summarised as Table 1.

Table 1: The Fundamental Equations of Investor Utility and Portfolio Composition

$$U_P = R_P - ½ .\sigma_P^2 \qquad (16)$$
$$[R_P - r_f] = \sigma_P^2 \qquad (20)$$
$$W = \Omega^{-1} R \qquad (21)$$
$$R_i = r_f + \beta_i * [R_M - r_f] \qquad (25)$$

For a "two risky assets (Stocks, Bonds) one risk-free asset" portfolio, equation 16 (with equations 13, 14 and 15) expands as:

$$U_P = \omega_S.(\mu_S + ½ \sigma_S^2) + \omega_B.(\mu_B + ½ \sigma_B^2) + (1- \omega_S - \omega_B).r_f - ½ c [\omega_S^2.\sigma_S^2 - ½ \omega_B^2.\sigma_B^2 - 2.C_{SB}. \omega_S.\omega_B. \sigma_S.\sigma_B] \qquad (12)$$

and equation 21 expands as:

$$R_S - r_f - (\omega_S. \sigma_S^2 + \omega_B. \sigma_{S,B}) = 0 \qquad (18)$$
$$R_B - r_f - (\omega_S. \sigma_{B,S} + \omega_B. \sigma_B^2) = 0$$

or:

$$R_S - r_f = \sigma_{S,P} \qquad (19)$$
$$R_B - r_f = \sigma_{B,P}$$

The implications of our above observations are summarised as follows::

1. Log-wealth utility investors: maximizing mean exponential return. For a log-wealth utility investor, maximizing a portfolio's utility reduces to choosing portfolio compositions ω_i so as to maximise the portfolio's mean exponential growth rate as represented by expression 12. The solution of which is determined by equations 21. Thus notwithstanding that the outcome utility offered by a portfolio is a function of the risk-return characteristics across the portfolio's component assets, the outcome utility is actually independent of the portfolio's overall risk as measured by the standard deviation, (σ_P) about the portfolio's mean exponential growth rate (μ_P).

2. Convolution of the CAPM. The set of equations 21 determine an investor's portfolio composition (ω_S and ω_B) in terms of the asset (S and B) returns, their variances and co-variances. The CAPM has been derived by refiguring the equilibrium conditions 21 with an asset's return R_i as the dependent variable. But we should be aware that an asset's beta (equation 24) itself relates linearly to the asset's outcome return possibilities: double an asset's outcome return possibilities, and we thereby double the covariance of such returns with the market's returns – and hence we double that asset's beta. And such new beta now works to "explain" the asset's higher returns in the first place.[9] Roll (1977) has drawn attention to this inherent circularity of the CAPM.

3. Idiosyncratic risk. In the CAPM (equation 25) the continuously compounded growth rate, R_i that generates the expected wealth outcome for asset i, may be represented (equation 13) as $R_i = \mu_i + ½ \sigma_i^2$, where σ_i represents that asset's total (market plus idiosyncratic) volatility about the stock's mean growth rate, μ_i. Thus the CAPM explained return has an idiosyncratic component. Malkiel and Xu (1997) observe that largely unpredicted sharp increases in idiosyncratic volatility occurred in US stocks over the period 1963-1994 of their study. They discover that portfolios on idiosyncratic risk from a low of 5.0 percent monthly volatility (σ_{min}) to 12.5 percent monthly volatility (σ_{max}) give rise to an annualised 7.0 percent return difference across such portfolios. Such difference actually understates the difference: $½[\sigma_{max}^2 - \sigma_{min}^2]*12 = ½[(0.125)^2 – (0.05)^2]*12 = 8.0$ percent (Dempsey, 2002).[10]

4 CONCLUSION

The paper has presented a binomial model for asset growth with volatility. Allowing a log-wealth utility, the model provides for the optimal allocation of assets in an investment portfolio. A feature of the model is that the equity risk premium is accounted for by the internal mathematics of risk over continuous time - rather than by the concept of investors setting prices at the commencement of a one-period investment time frame as in the CAPM. In which context, portfolio idiosyncratic risk is recognised as a component of returns. Notwithstanding, the formulation is actually consistent with the CAPM in continuous time. It has been observed that the model's predictions fit quite well with Malkiel and Xu's (1997) empirically observed relationship between US equity performances and idiosyncratic risk.

It is noted that the model is consistent with the framework developed by Merton (1969, 1973) and with Samuelson's arguments (1963, 1989, 1994) that portfolio choices can be indifferent to the investor's time-horizon. It is also consistent with Dempsey's (2002) model applied to a single risky asset combined with a risk-free asset.

NOTES

1. The assumption that stock market growth rates are normally distributed is broadly justified by the evidence of past stock price performance (for example, Fama, 1976, ch. 2; and more recently, Jones and Wilson, 1999); while Fama (1976) and Ibbotson Associates (2001) observe that an *a priori* expectation for a normal distribution is reinforced by the mathematics of selection as captured by the Central Limit Theorem.

2. Hence the mean continuously compounded return grows linearly with time while the standard deviation about such return grows as the square-root of time.

3 For example. we have on comparing across Figures 1 and 2, the first component of the operator as: $\tfrac{1}{4}(1+C_{SB})\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} = \tfrac{1}{4}(1+C_{SB})\ exp(\mu_0 + \sigma_0)$, and the final component of the operator as: $\tfrac{1}{4}(1+C_{SB})\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)\} = \tfrac{1}{4}(1+C_{SB})\ exp(\mu_0 - \sigma_0)$. Taking logs of both sides of the two equations and adding yields equation 1, with similar manipulations for equations 2 – 4.

4. For example, the third listed wealth outcome in column 3, namely, $1/8\ (1+C_{SB}). \quad (1-C_{SB}).exp(\mu_0 + \sigma_0 + \mu_1 - \sigma_1)$ {1, 3}, is the outcome of the probability-weighted operator: $\tfrac{1}{4}\ (1+C_{SB}).exp(\mu_0 + \sigma_0)$ {1} either followed or preceded by the probability-weighted operator: $\tfrac{1}{4}\ (1-C_{SB}).exp(\mu_1 - \sigma_1)$ {3} (as depicted by the arrows in Figure 2).

5. A certain plausibility argument can be made for the natural log-wealth utility function in continuous time. To see this, consider that the continuously applied growth rate r followed by the rate $-r$ for equal durations, effectively cancel. Thus it appears reasonable to stipulate that the positive utility $U(r)$ of the continuously applied growth rate r should equate with the negative utility $U(-r)$ of the same continuously applied growth rate $-r$. Now, set r^* as the benchmark utility for some continuously compounded growth rate r^* (so that $U(r^*) = r^*$ for such r^*). If now we allow that $U(2r^*)$ does not necessarily equal $2r^*$, let us say, is less than $2r^*$, we are obliged to concede that $U(-2r^*)$ is greater (ie, less negative) than $-2r^*$. In which case we have the seemingly implausible outcome that investors are subject to decreasing added utility with each additional unit of wealth accumulation, while remaining subject to decreasing subtracted utility with each additional unit of wealth erosion.

6. The utility of the outcomes in column 2 of Figure 2 (U_P) is determined as: $U_P = \tfrac{1}{4}(1+C_{SB}).\ln[exp(\mu_0 + \sigma_0)] + \tfrac{1}{4}(1-C_{SB}).\ln[exp(\mu_1 + \sigma_1)] + \tfrac{1}{4}(1-C_{SB}).\ln[exp(\mu_1 - \sigma_1)] + \tfrac{1}{4}(1+C_{SB}).\ln[exp(\mu_0 - \sigma_0)] = \tfrac{1}{2}\ [(1+C_{SB}).\mu_0 + (1-C_{SB}).\mu_1]$, as in the text.

7. Standard deviations reduce as the inverse of the square-root of time whereas continuously compounded returns reduce as the inverse of time (see footnote 2). Hence it is appropriate to allow terms as one order higher for standard deviation than for continuously compounded returns.

8. Our result could have been obtained equally by differentiating equation 10 with respect to ω_S and ω_B and setting the outcomes equal to zero. Differentiating with respect to ω_S we have:

$$\frac{(1+C_{SB})\,.\,[exp(\mu_S + \sigma_S) - exp(r_f)]}{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)}$$

$$+ \frac{(1-C_{SB})\,.\,[exp(\mu_S + \sigma_S) - exp(r_f)]}{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)}$$

$$+ \frac{(1-C_{SB})\,.\,[exp(\mu_S - \sigma_S) - exp(r_f)]}{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)}$$

$$+ \frac{(1+C_{SB})\,.\,[exp(\mu_S - \sigma_S) - exp(r_f)]}{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).exp(r_f)}$$

$$= 0$$

which follows on observing that when $y = \ln[f(x)]$, where $f(x)$ is a continuous function of x, the partial differentiation of y with respect to x is determined as: $\delta y/\delta x = [\delta f(x) / \delta x] / f(x)$.

Allowing the relationship $exp(y) = 1 + y + \tfrac{1}{2}y^2$, for each continuously compounded function over a sufficiently short time period, while restricting the analysis to terms of order μ_S, μ_B, r_f, σ_S^2, σ_B^2 (footnote 7) the above equation is expressed:

$$\begin{aligned}
&[\,(1+C_{SB})\,.\,[\,\mu_S + \sigma_S + \tfrac{1}{2}\,\sigma_S^2 - r_f - \omega_S.\,\sigma_S^2 - \omega_B.\,\sigma_S\ \sigma_B\,]\\
+\ &[\,(1-C_{SB})\,.\,[\,\mu_S + \sigma_S + \tfrac{1}{2}\,\sigma_S^2 - r_f - \omega_S.\,\sigma_S^2 + \omega_B.\,\sigma_S\ \sigma_B\,]\\
+\ &[\,(1-C_{SB})\,.\,[\,\mu_S - \sigma_S + \tfrac{1}{2}\,\sigma_S^2 - r_f - \omega_S.\,\sigma_S^2 + \omega_B.\,\sigma_S\ \sigma_B\,]\\
+\ &[\,(1+C_{SB})\,.\,[\mu_S - \sigma_S + \tfrac{1}{2}\,\sigma_S^2 - r_f - \omega_S.\,\sigma_S^2 - \omega_B.\,\sigma_S\ \sigma_B\,]\\
=\ &0
\end{aligned}$$

which yields:

$$\mu_S + \tfrac{1}{2}\,\sigma_S^2 - r_f - \omega_S.\,\sigma_S^2 - \omega_B.\,C_{SB}.\sigma_S.\sigma_B = 0$$

On identifying (i) $R_S = \mu_S + ½\, \sigma_S^2$ (equation 13) and (ii) the covariance between assets stocks and bonds as $\sigma_{S,B} = C_{SB}.\sigma_S .\sigma_B$ (equation 17), we can write the above equation as:

$$R_S - r_f \quad - \quad \omega_S.\, \sigma_S^2 \quad - \quad \omega_B.\, \sigma_{S,B} \quad = \quad 0$$

Similarly on differentiating equation 10 with respect to ω_B, we obtain:

$$R_B - r_f \quad - \quad \omega_B.\, \sigma_B^2 \quad - \quad \omega_S.\, \sigma_{B,S} \quad = \quad 0$$

which are equations 18 of the text.

9. There are potentially significant implications for traditional measures of portfolio management performance here. Suppose, for example, we have two stocks which offer identical claims on a firm, but that the price of one stock has remained and is expected to remain half that of the other. It follows that the potential dividend yields (and hence we assume returns) for the lower-priced stock are twice as for the higher-priced stock. The standard deviation of potential returns – and hence beta - for the lower-priced stock are therefore twice as for the higher-priced stock. Consider now that a portfolio manager A provides his client with a portfolio of the low-priced stocks, while portfolio manager B provides her client with a portfolio of (half as many) of the high-priced stocks (for the same investment outlay). Clearly portfolio manager A has the better performance. Notwithstanding, standard CAPM performance ratios, for example, Treynor's ratio as:

$$T = \frac{\text{[portfolio return – risk-free deposit rate]}}{\text{portfolio beta}} = \frac{R_P - r_f}{\beta_P}$$

and Sharpe's ratio as:

$$S = \frac{\text{[portfolio return – risk-free deposit rate]}}{\text{total portfolio volatility}} = \frac{R_P - r_f}{\sigma_P}$$

with (to make the point) an r_f effectively close to zero, are identical across both lower- and higher-priced stock portfolios, with the outcome that the two portfolio managers are pronounced as having performed equally for their clients on a risk-return basis.

10. The portfolio's periodic arithmetic return (AR_P) – calculated for a sequence of N (equally) discrete periodic returns (r_i) as:

$$AR_P = 1/N \sum_{i=1}^{N} r_i$$

[where each $r_i = W_i/W_{i-1} - 1$, W_i = outcome wealth at end of period i] - and the standard deviation about such return (S_P) as measured by Malkiel and Xu do not strictly identify the portfolio's mean continuous growth rate (μ_P) and the standard deviation about such rate (σ_P). The theoretical relationships are: $\mu_P = \ln(1+AR_P) - ½ \ln\{1 + [S_P/(1+AR_P)]^2\}$; and $\sigma_P^2 = \ln\{1 + [S_P/(1+AR_P)]^2\}$(for example, de la Grandville, 1998, Ibbotson Associates, 2001, Jacquier, Kane and Marcus, 2003).

APPENDIX : FIGURES

Figure 1:

The binomial probability-weighted outcomes of investing $1 in risky stocks, risky bonds and a risk free asset in proportions: ω_S: ω_B: (1- ω_S - ω_B).

Starting wealth	Wealth after one period
$1	$¼\,(1+C_{SB}).\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).\ exp(r_f)\}$
	$¼\,(1-C_{SB}).\{\omega_S.exp(\mu_S + \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).\ exp(r_f)\}$
	$¼\,(1-C_{SB}).\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B + \sigma_B) + (1- \omega_S - \omega_B).\ exp(r_f)\}$
	$¼\,(1+C_{SB}).\{\omega_S.exp(\mu_S - \sigma_S) + \omega_B.exp(\mu_B - \sigma_B) + (1- \omega_S - \omega_B).\ exp(r_f)\}$

Figure 2:

The binomial probability-weighted outcomes of investing \$1 in risky stocks, risky bonds and a risk free asset in proportions: ω_S: ω_B: (1- ω_S - ω_B) over two periods.

-1- Starting Wealth	-2- Wealth after One Period	-3- Wealth after Two Periods	
		1/16 $(1+C_{SB})^2$ $exp(2\mu_0 + 2\sigma_0)$	{1, 1}
	¼ $(1+C_{SB})$ $exp(\mu_0 + \sigma_0)$ {1}	1/8 $(1+C_{SB})(1-C_{SB})$ $exp(\mu_0 + \sigma_0 + \mu_1 + \sigma_1)$	{1 2}
		1/8 $(1+C_{SB})(1-C_{SB})$ $exp(\mu_0 + \sigma_0 + \mu_1 - \sigma_1)$	{1 3}
	¼ $(1-C_{SB})$ $exp(\mu_1 + \sigma_1)$ {2}	1/8 $(1+C_{SB})^2$ $exp(2\mu_0)$	{1, 4}
		1/16 $(1-C_{SB})^2$ $exp(2\mu_1 + 2\sigma_1)$	{2, 2}
\$1		1/8 $(1-C_{SB})^2$ $exp(2\mu_1)$	{2, 3}
	¼ $(1-C_{SB})$ $exp(\mu_1 - \sigma_1)$ {3}	1/8 $(1+C_{SB})(1-C_{SB})$ $exp(\mu_0 - \sigma_0 + \mu_1 + \sigma_1)$	{2, 4}
		1/16 $(1-C_{SB})^2$ $exp(2\mu_1 - 2\sigma_1)$	{3, 3}
	¼ $(1+C_{SB})$ $exp(\mu_0 - \sigma_0)$ {4}	1/8 $(1+C_{SB})(1-C_{SB}).exp(\mu_0 - \sigma_0 + \mu_1 - \sigma_1)$	{3, 4}
		1/16 $(1+C_{SB})^2$ $exp(2\mu_0 - 2\sigma_0)$	{4, 4}

Mean Exponential growth rate:

μ_P $2\,\mu_P$

$= \frac{1}{2}\,[(1+C_{SB}).\mu_0 + (1-C_{SB}).\mu_1]$

Utility of investment

μ_P $2\,\mu_P$

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