Positive Linear Dynamic Model of Mobile Source Air Pollution and Related Problems

L Caccetta[a], S. Kostova[b] and V Rumchev[a]

[a] Department of Mathematics and Statistics, Curtin University of Technology, GPO Box U1987, Perth, Western Australia, Australia.

[b] Institute of Control and System Research, Bulgarian Academy of Sciences, P.O.B. 79, 1113, Sofia, Bulgaria

Abstract : Mobile Source Air Pollution (MSAP) emissions as a total percentage of total air emissions have dramatically increased throughout the world over the past decade. The CO, NO_x and HCs components of these emissions create air quality problems and are considered as a major contributor to the development of lung cancer and various respiratory diseases. In this paper, we build a dynamic model of MSAP based on the theory of positive systems. Our model is capable of predicting emissions of the total vehicle population. Further, it provides tools for maintaining allowable levels of emissions.

Keywords **:** *Environmental health; mobile source air pollution; positive dynamic systems; optimisation.*

1. INTRODUCTION

In this paper a mobile source air pollution (MSAP) model is presented and related problems are defined and solved. MSAP as a percentage of total air emissions has dramatically increased over the past few decades. For example (Deaton and Winebreak (2000)) in the USA in 1995 highway vehicles contributed 64% of national CO emissions, 35% of national NO_x emissions and 27% of national volatile organic compound (VOC) emissions. Millions of vehicles (automobiles, trucks, buses etc.) are on the road and most of them operating four-stroke, spark ignition internal combustion engine (ICE) technology. The components of the exhaust gas such as CO, NO_x and HC's create air quality problems in urban areas throughout the world. These components, as well as some others are a major cause of lung cancer and a variety of respiratory illnesses.

The purpose of this paper is to build a dynamical model of mobile source air pollution in order to predict emissions of the total vehicle population, and to formulate and solve the problems of maintaining acceptable levels of emissions. We illustrate our method with a numerical example.

2 MODEL DESCRIPTION

Throughout this paper for a general matrix $A = [a_{ij}]_{n\times s}$, we write $A \geq 0$ (or $A \in R_+^{n\times s}$) if $a_{ij} \geq 0$ for all i,j; $A > 0$ if at least some entry $a_{ij} > 0$ and $A >> 0$ if $a_{ij} > 0$ for all i,j.

Consider a vehicle population on the road in a given area (state, city, etc.). The total vehicle population can be divided into *n* different cohorts (categories, groups) on the basis of their "age". Let $C_i, i = 1,2,\ldots,n$, represent the cohort of vehicles of age in the interval $(a_i^l, a_i^u]$, $i = 1,2,\ldots,n-1$, where a_i^l and a_i^u are, respectively, lower and upper limits of the vehicles age in the cohort C_i. T_i is the number $(a_i^u - a_i^l)$ for $i = 1,2,...,n-1$ and the last cohort C_n consist of all the vehicles of age greater than $T_n = \sum_{i=1}^{n-1} T_i$.

Let $x_i(t) \geq 0$, $t = 1,2...$, denote the number of vehicles of cohort C_i at the beginning of time period *t* (normally a year) and $x_i(0), i = 1,\ldots,n$, the initial number of vehicles in each cohort. It is assumed, in general, that during one time period (*t*, *t*+1) a fraction of the vehicles constituting cohort C_i will progress to cohort C_{i+1}, a fraction will remain in C_i and a fraction will not survive. We define survival rates $s_i, i = 1,\ldots,n$, as the fraction of survived vehicles (not scrapped because of long lifetime, or due to accidents or breakdowns) while in the cohort. Then the fraction of "disposed" vehicles is equal to $(1 - s_i), i = 1,\ldots,n$. In this model it is also assumed that at the beginning of time period *t* the purchase of vehicles (new and used) from outside the system is used as control variables $u_i(t) \geq 0, i = 1,\ldots,p;\quad t = 0,1,2,\ldots$.

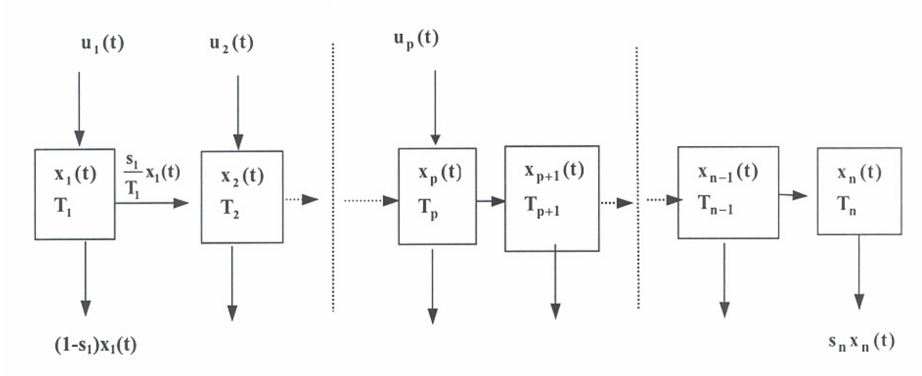

Table : Start and Finish Times

Let $m_i, i = 1,\ldots,n$ be the average annual distances (in *km* or *miles*) travelled by one vehicle of cohort C_i and e_{ji} be the average emission (in *g/km* or *g/miles*) of pollutant type $j, j = 1,2,\ldots,k$ from one vehicle of cohort C_i (emission factor). Note that $e_{ji} > 0$ for all i,j. Then we can calculate total emission $\varepsilon_j(t)$ of type *j* at the time period *t* by using the following equation:

$$\varepsilon_j(t) = \sum_{i=1}^{n} x_i(t) m_i e_{ji}, \; j = 1,2,\ldots,k; \quad \varepsilon_j(t) > 0 \quad (1)$$

Let us suppose that the survival rates, emission factors and average annual distances are known from demographic and engineering studies and that there are accepted standards for the emissions from mobile sources. We will introduce these standards as upper limits $\bar{\varepsilon}_j, j = 1,2,\ldots,k$ of total pollution of type *j,* i.e. the total emissions are subject to the following restriction:

$$\varepsilon_j(t) \le \bar{\varepsilon}_j, j = 1,2,\ldots,k \,. \quad (2)$$

The block diagram of the model is given in a Fig. 1.

3 SYSTEM DYNAMICS

Taking into account the above notation, the dynamics of the system may be described by the system of difference balance equations:

$$x_1(t+1) = s_1 x_1(t) - \frac{1}{T_1} s_1 x_1(t) + u_1(t) \,,$$

$$x_i(t+1) = s_i x_i(t) + \frac{1}{T_{i-1}} s_{i-1} x_{i-1} - \frac{1}{T_i} s_i x_i(t)$$

$$+ u_i(t), i = 2,\ldots,n-1$$

$$x_n(t+1) = s_n x_n(t) + \frac{1}{T_{n-1}} s_{n-1} x_{n-1}(t) \,.$$

The above can be expressed in the matrix-vector form:

$$x(t+1) = Ax(t) + Bu(t), t = 0,1,2,\ldots \quad (3)$$

with $x(t)$ the state of the system at time *t*, $u(t)$ the control (decision) vector and system matrices A, B having the following structure:

$$A = \begin{bmatrix} s_1(1-\frac{1}{T_1}) & 0 & 0 & \cdots & 0 \\ \frac{s_1}{T_1} & s(1-\frac{1}{T_2}) & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \frac{s_{n-2}}{T_{n-2}} & s_{n-1}(1-\frac{1}{T_{n-1}}) & 0 \\ 0 & 0 & 0 & \frac{s_{n-1}}{T_{n-1}} & s_n \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (4)$$

The mobile source air pollution "cohort-type" model built above is a positive linear discrete-time dynamic model. The reachability and controllability properties of cohort-type models are discussed in (James and Rumchev (2000)). The system matrices have several structural properties:

- The matrix ***A*** is nonnegative and

  $$\sum_{i=1}^{n} a_{ij} = s_i < 1$$

  implies that $\rho(A) < 1$ where ρ is the spectral radius of ***A*** (Luenberger (1979)).

- The matrix ***B*** is a non-negative monomial matrix (with linearly independent columns having exactly one non-zero entry).

The theory of positive linear discrete time systems (Farina and Rinaldi (2000) and Luenberger (1979)), can be applied to study and improve the behavior of the MSAP system. On the basis of this model we formulate and solve some problems in the following section.

4 RELATED PROBLEMS

We illustrate the application of the above model by detailing four specific problems that are of interest in the context of air pollution control. We have:

1. Emission factor matrix $E = [e_{ji}]_{k\times n};\ E >> 0$
2. Average travelled distances matrix $M = \mathrm{diag}(m_1, m_2, \dots m_n)$, where m_i is the average distance travelled by cohort C_i vehicles.
3. Emission constraints vector $\bar{\varepsilon} = [\,\bar{\varepsilon}_j], j = 1,2,\dots,k;\ \bar{\varepsilon} >> 0$.
4. The PLDS system, described by the equation (3) with state and control matrices having the structure given in (4).
5. Vector initial state $x(0) = x_0$.

Problem 1 **: Predicting the future emission levels**

If we have data for the vehicle purchases $u(0), u(1), \dots, u(t-1)$ and taking into account (1) the total MSAP $\varepsilon(t)$ in time period *t* can be calculated by using matrix equation:

$$\varepsilon(t) = EMx(t), \qquad (5)$$

where

$$\begin{aligned} x(t) &= A^t x(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) \\ &\quad + \dots + ABu(t-2) + Bu(t-1) \qquad (6) \\ &= A^t x_0 + \Re_t u_t, \end{aligned}$$

where $\Re_t$ is the *t*-step reachability matrix, $\Re_t(A,B) = \left[B \mid AB \mid A^2B \dots \mid A^{t-1}B\right]$ and u_t is the extended control vector,

$$u_t = [u(t-1)\dots u(0)]^T, t = 1,2,\dots,\ u_k \in R_+^{tp}.$$

An important problem from both a practical and theoretical point of view is the problem of maintaining acceptable levels of emissions in each time period. For this purpose we define the problems of one-step maintainability and *T*-step maintainability.

Problem 2**:One-step maintainability problem**

For a given system (1-5) with $EMx(0) \le \bar{\varepsilon}$, *find a decision vector (vehicles purchased in the system) ensuring that the levels of emissions after one time period will be within the allowed limits.*

By using (5) we can calculate the total emissions after one time period. Taking into account (2) we have

$$EM(Ax(0) + Bu(0)) \le \bar{\varepsilon}.$$

Substituting $EM = E_M$, $x_0 = x(0), u_0 = u(0)$ and after elementary matrix calculations we have:

$$E_M Bu_0 \le \bar{\varepsilon} - E_M Ax_0.$$

Taking into account the structure of the matrix B given in (4), we have:

$$E_{Mp}u_0 \le \bar{\varepsilon} - E_M Ax_0, \qquad (7)$$

where the matrix E_{Mp} consist of the first *p* columns of the matrix E_M.

The general solution to the system of linear inequalities (7), or equivalently the dual representation (in terms of vertices) of the polytope (7) gives the set of all admissible purchases of vehicles, which maintain the emissions within the prescribed standard values. One can introduce a meaningful criterion and formulate a related linear programming problem to determine the optimal policy of purchase of vehicles:

$$\max z = b^T u(0)$$

subject to

$$\begin{aligned} &E_{Mp}u_0 \le \bar{\varepsilon} - E_M Ax_0 \qquad (8) \\ &u_0 \ge 0 \\ &b_i \ge 0, i = 1,\dots,p;\ \sum_i b_i = 1, \end{aligned}$$

where $b_i, i = 1,\dots,p$ are weighted coefficient representing the importance of a variety of factors in the decision-making process. For example, $b_1 = 1$ and $b_i = 0, i = 2,\dots,p$ if only new vehicles are sold on the market.

Problem 3**: T-step maintainability problem**

For a given system (1-5) find a decision vector (vehicles purchased into the system) ensuring that the levels of emissions after T *time periods will be within the allowed limits.*

We can calculate the total emission after T time periods by using:

$$x(1) = Ax(0) + Bu(0)$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) = A^2x(0) \\ &\quad + ABu(0) + Bu(1) = A^2x(\mathbf{0}) + \Re_2 u_2 \end{aligned}$$

$$\begin{aligned} x(3) &= Ax(2) + Bu(2) = A^3x(0) \\ &\quad + A^2Bu(0) + ABu(1) + Bu(2) \\ &= A^3x(0) + \Re_3 u_3 \end{aligned}$$

…………………………………………..

$$x(T-1) = Ax(T-2) + Bu(T-2)$$
$$= A^{T-1}x(0) + A^{T-2}Bu(0) + \ldots + Bu(T-2)$$
$$= A^{T-1}x(0) + \Re_{T-1}u_{T-1}$$

where $\Re_i$ and $u_i, i = 1, \ldots, T$ are as it is given in (6). Then denoting $x(0) = x_0$ and taking into account (5) and (2), for the considered time period we have:

$$E_M x(1) \le \bar{\varepsilon}, E_M \times (2) \le \bar{\varepsilon}, E_M x(3) \le \bar{\varepsilon}, \ldots,$$
$$E_M x(T-1) \le \bar{\varepsilon},$$

or equivalently :

$$\begin{aligned} E_M Bu_1 &\le \bar{\varepsilon} - E_M Ax_0 \\ E_M \Re_2 u_2 &\le \bar{\varepsilon} - E_M A^2 x_0 \\ E_M \Re_3 u_3 &\le \bar{\varepsilon} - E_M A^2 x_0 \\ &\ldots\ldots \\ E_M \Re_{T-1} u_{T-1} &\le \bar{\varepsilon} - E_M A^{T-1} x_0 . \end{aligned} \qquad (9)$$

Note that
$u(k) \in R_+^m, k = 0,1,..., T-2,$, and the extended control vector $u_t = [u_t(t-1), \ldots, u(1), u(0)]^T$.

The general solution to the system of linear inequalities (9) according to the elements of the extended control vector gives the set of all admissible purchases of vehicles, which maintain the emissions within the prescribed standard values during T-time periods. One can introduce a meaningful criterion and formulate a related linear programming problem to determine the optimal policy of purchase of vehicles in T- time periods.

max $z = b^T u_{T-1}$

subject to

$$\begin{aligned} E_M Bu_0 &\le \bar{\varepsilon} - E_M Ax_0 \\ E_M \Re_2 u_2 &\le \bar{\varepsilon} - E_M A^2 x_0 \\ E_M \Re_3 u_3 &\le \bar{\varepsilon} - E_M A^2 x_0 \\ &\ldots\ldots \\ E_M \Re_{T-1} u_{T-1} &\le \bar{\varepsilon} - E_M A^{T-1} x_0 \\ u_{T-1} &\ge 0 \end{aligned}$$

***Problem 4:* Feedback holdability problem**

Using (2) and (5) we can obtain the admissible set in the state space. By using the substitution $EM = E_M$, $E_M >> 0$, the matrix inequality (2) is transformed to

$$E_M x(t) \le \bar{\varepsilon} . \qquad (10)$$

The inequality (10) describes an admissible set

$$P = \{x \mid E_M x \le \bar{\varepsilon}\} \qquad (11)$$

in the state space. The set P is polytope (a bounded polyhedral set) and can be represented in terms of its vertices $v_1, \ldots, v_N$ namely

$$P = \mathrm{conv}(v_1, \ldots, v_m) = \mathrm{conv}(V)$$
$$= \{x \in R_+^n \mid x = \sum_{i=1}^{N} \alpha_i v_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \ge 0\}, \qquad (12)$$

where $V \in R^{n \times N}$ is the matrix whose column are the vertices $v_1, \ldots, v_N$ of ***P***.

The problem of maintaining acceptable levels of emissions will be solved if we can make the polytope positively invariant. We can do that by solving the feedback holdability problem (Rumchev (2000)) for the system (3).

Feedback holdability problem: For the system (3) with admissible set (11) and initial state $x(0) = x_0 \in P$, find a linear nonnegative control law

$$u(t) = Fx(t), F \in R_+^{n \times p} \qquad (13)$$

such that the set P is positively invariant with regard to the motion of the closed-loop system

$$x(t = 1) = (A + BF)x(t) = A_c x(t) . \qquad (14)$$

Let polytope P consisting of all subconvex combinations of vectors $v_1, ..., v_n$ be denoted by

$$P = \mathrm{subconv}(v_1, \ldots, v_N)$$
$$= \left\{ x \in R^n \middle| x = \sum_{i=1}^{N} \alpha_i v_i, \sum_{i=1}^{N} \alpha_i \le 1, \alpha_i \ge 0 \right\}.$$

We will give a lemma (Benvenuti and Farina (2002)) that will be used in the sequel.

Lemma: Assume $0 \in \mathrm{conv}(v_1, \ldots, v)$. Then $\mathrm{conv}(v_1, \ldots, v_N) \equiv \mathrm{subconv}(v_1, \ldots, v_N)$.

The proof of the following result is based on the above Lemma (Caccetta, et.al (2003)).

Theorem: The polytope $P = \mathrm{conv}(V)$, with $0 \in P$, is positively invariant with regard to the motions of the closed loop system (14) if and only if there exists a quasistochastic matrix $S = [s_{ij}]_{N \times N}$ $\left(\sum_{i=1}^{N} s_{ij} \le 1 \right)$ such that:

$$(A + BF)V = VS. \qquad (15)$$

Taking into account the above Theorem the Feedback holdability problem can be solved as linear programming problem as follows:

LP Problem:

max $b^T Fe_n$

subject to

$$(A + BF)V = VS$$

$$F \geq 0, S \geq 0$$
$$S^T e_N \leq 1$$

where vectors e_n and e_N have all elements equal to one and vector b is weighted vector representing the importance of a variety of factors in decision-making process.

5 EXAMPLE

We will now illustrate the above problems with a numerical example. Our example modifies that presented in Deaton and Winebrake (2000). The modification involves an additional emission to control, the introduction of an emission standards and a possibility for vehicle purchases in first two age groups. Let the following be given:

1. Emission factor matrix, emissions measured in *g/mile*
$$E = \begin{bmatrix} 3 & 3.4 & 3.8 & 4.2 & 5 \\ 1 & 1.5 & 2 & 2.5 & 3 \end{bmatrix}.$$
Average travelled distances matrix, distances measured in *miles*
$M = \text{diag}(12000, 12000, 12000, 12000, 12000)$
2. Emission constraints vector (g) is
$$\bar{\varepsilon} = \begin{bmatrix} 1.4\ 10^{10} \\ 0.7\ 10^{10} \end{bmatrix}.$$
3. Survival rates

$s_1 = 0.90$; $s_2 = 0.85$; $s_3 = 0.80$; $s_4 = 0.75$; $s_5 = 0.60$

4. Cohort ages: 0-2years, 3-5 years, 6-8 years, 9-11 years and >11 years.

5. Initial state is
$x(0) = [100000 \quad 100000 \quad 75000 \quad 75000 \quad 50000]^T$

System matrices are:

$$A = \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0.5667 & 0 & 0 & 0 \\ 0 & 0.2833 & 0.5333 & 0 & 0 \\ 0 & 0 & 0.2667 & 0.5 & 0 \\ 0 & 0 & 0 & 0.25 & 0.6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 1 Consider prognosed data for vehicle purchases for 10 years ahead as follows:

$$u_{10} = \begin{bmatrix} 20000 & 20000 & 20000 & 40000 & 40000 & 20000 & 20000 & 20000 & 20000 & 20000 \\ 15000 & 15000 & 15000 & 30000 & 30000 & 15000 & 15000 & 15000 & 15000 & 15000 \end{bmatrix}$$

Then by using (6) we can obtain the emissions, which are plotted at Fig.2

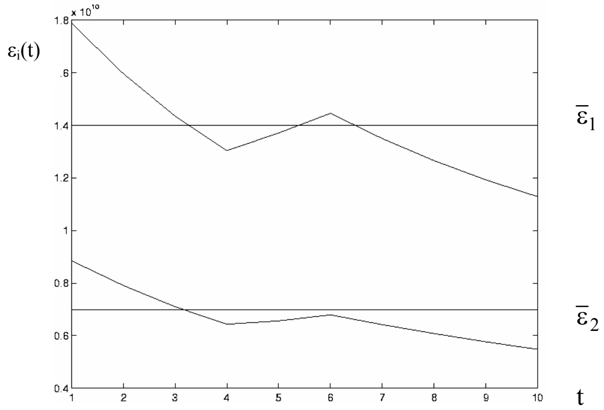

Fig.2 Predicted Emission Levels

It is obvious from for the above graphics, that at the moment *t*=6, the first emission exceed admissible level. So by solving the one-step maintainability problem at the beginning of 5-th year we will obtain an admissible control.

One-step maintainability

The initial state (prognosed vehicles at the beginning of 5th year) is: $x(5) = [82480 \quad 102323 \quad 58700 \quad 37023 \quad 32972]^T$.

We solve the following LP problem:

max $z = b_1 u_1 + b_2 u_2$

subject to

$$0.36 u_1 + 0.41 u_2 \leq 26293,$$
$$0.12 u_1 + 0.18 u_2 \leq 13781$$
$$u_1 \geq 0, u_2 \geq 0.$$

For the weighted vector: $b = [1 \quad 0]^T$ the decision is $u_1 = 73037, u_2 = 0$ and

$x(6) = [122520 \quad 82730 \quad 60300 \quad 34160 \quad 29040]^T$.

For the weighted vector: $b = [0 \quad 1]^T$ the decision is $u_1 = 0, u_2 = 64444$ and

$x(6) = [49490 \quad 147170 \quad 60300 \quad 34160 \quad 29040]^T$

In the both cases the emissions at the beginning of 6-th year are within the allowable limits.

T-step maintainnability

Consider the MSAP system at an initial state

$x(0) = [60800 \quad 90146 \quad 62173 \quad 40886 \quad 37918]^T$, belonging to the admissible state region. We have to obtain the control strategy ensuring that for three years ahead $E_M x(t) \le \bar{\varepsilon}$, $t = 1,2,3$. Let the unknown 3-step control vector be

$$u = [u_1^{(0)} \quad u_2^{(0)} \quad u_1^{(1)} \quad u_2^{(1)} \quad u_1^{(2)} \quad u_2^{(2)}]^T,$$

where

$$u^{(0)} = [u_1^{(0)} \quad u_2^{(0)}]^T, \; u^{(1)} = [u_1^{(1)} \quad u_2^{(1)}]^T,$$

$$u^{(2)} = [u_1^{(2)} \quad u_2^{(2)}]^T$$

are respectively the controls at the beginning of first, second and third years. Using (9) we define the following LP problem:

$$\mathbf{max}\, z = b^T u$$

subject to

$$E_M B u^{(0)} \le \bar{\varepsilon} - E_M A x_0$$

$$E_M A B u^{(0)} + E_M B u^{(1)} \le \bar{\varepsilon} - E_M A^2 x_0$$

$$E_M A^2 B u^{(0)} + E_M A B u^{(1)}$$

$$+ E_M B u^{(2)} \le \bar{\varepsilon} - E_M A^3 x(0)$$

$$u(2) \ge 0$$

By using the weighted vector

$$b = [0.333 \quad 0 \quad 0.333 \quad 0 \quad 0.333 \quad 0]^T$$

we obtain the following control sequence

$$u = [92703 \quad 0 \quad 62170 \quad 0 \quad 58740 \quad 0]^T$$

and by using the weighed vector

$$b = [0 \quad 0.333 \quad 0 \quad 0.333 \quad 0 \quad 0.333]^T$$

we obtain the control sequence

$$u = [0 \quad 81797 \quad 0 \quad 59493 \quad 0 \quad 59232]^T,$$

both ensuring three years allowable emissions.

Feedback holdability problem

Consider the MSAP dynamic system with:

$$A = \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 3 & 3.4 \\ 1 & 1.5 \end{bmatrix}$$

$$M = \begin{bmatrix} 12000 & 0 \\ 0 & 12000 \end{bmatrix} \quad \bar{\varepsilon} = \begin{bmatrix} 1.4\,10^{10} \\ 0.7\,10^{10} \end{bmatrix}.$$

The admissible set in the state space is a polytope, represented by the inequalities:

$$0.36x_1 + 0.12x_2 \le 140000$$

and $0.12x_1 + 0.18x_2 \le 70000$, and its vertices are columns of the matrix

$$V = \begin{bmatrix} 0 & 388890 & 0 & 333330 \\ 0 & 0 & 388890 & 166670 \end{bmatrix}.$$

When the weighed vector chosen is

$$b = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$$

we obtain the feedback matrix

$$F = \begin{bmatrix} 0.0001 & 0.4676 \\ 0.0004 & 0 \end{bmatrix} 10^{-4}$$

and the quasistochastic matrix

$$S = \begin{bmatrix} 0.5570 & 0.1072 & 0.1569 & 0 \\ 0.0001 & 0.5999 & 0.0005 & 0.5144 \\ 0.0001 & 0.3 & 0.7999 & 0.6 \\ 0.0001 & 0 & 0.0001 & 0 \end{bmatrix},$$

ensuring the invariance of the admissible set in the state space.

6 CONCLUSIONS

By using the theory of positive dynamic systems a model for mobile source air pollution is developed. The MSAP dynamic model and the related problems considered in this paper provide tools for maintaining allowable levels of emissions from mobile vehicles in urban areas. Our method can be immediately used to support the decision making process of regional and state environmental authorities.

7 ACKNOWLEDGEMENTS

This work is financially supported by an Australian Research Council Large Grant A00001126.

8 REFERENCES

Benvenuti L. and Farina L., Linear programming approach to constrained feedback control, Int. J. of Systems Science, vol. 33, N1, pp. 4 5-53, 2002.

Caccetta L., Kostova S. and Rumchev V., A Positive Linear Dynamic Model of Mobile Source Air Pollution. (preprint 2003).

Deaton M.L. and Winebrake J.J., Dynamic modelling of environmental systems, Springer-Verlag, 2000.

Farina L. and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, Wiley, New York, 2000.

James D.J.G. and Rumchev V.G., Cohort-type models and their reachability and controllability properties, Systems Science, vol.26, No2, 2000.

Luenberger D., Introduction to dynamic systems theory, models and application, Wiley & Sons, New York 1979.

Rumchev V.G., Feedback and positive feedback holdability of discrete-time systems, Systems Science, vol. 26, No.3, 2000.