Testing for a Single Factor Model in the Multivariate State Space Framework

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EXTENDED ABSTRACT

The dynamic factor model proposed by Stock and Watson (1989) has been widely used in that it can express the behaviour of a large set of variables using a small number of factors. This model has been especially useful in macro economics and financial econometrics. See Stock and Watson (2005) for more recent development of dynamic factor models. As far as I know, most of research has addressed the estimation of the model, and relatively less attention has been paid to hypothesis testing.

We here consider the problem of deciding the number of factor. This problem was considered only by Bai and Ng (2002) using an information criterion in panel analysis of large cross-sections and large time dimensions. Unlike theirs, this paper considers this problem in the framework of the bivariate time series analysis ; we only assume that the time series data is long enough. Instead, we have to specify more detailed structure of the dynamic process. We propose the Lagrange multiplier test for the hypothesis that two variables have a single common dynamic factor; the result can be easily generalized to the case of more than two variables. The null hypothesis of the paper is that the variables have a single common factor and the null is defined by equal autoregression coefficients and perfectly correlated disturbances. The Lagrange multiplier test, which requires only the estimation under the null hypothesis, is useful, because the Wald and likelihood ratio tests require estimation of the large number of factors under the alternative hypothesis.

The dynamic factor model is estimated by the linear Kalman filtering. We obtain the formula of the test statistic using the derivative formula of degenerate density in integrals. It is shown that the test statistic is obtained from the conditional expected value and covariance of the dynamic factor. This method is practical only when the number of variable is not too many, because the covariance matrix is obtained by matrix inversion. However, we suppose that this method is applicable to problems of practical size on account of the recent advancement of computing power

1. INTRODUCTION

The dynamic factor model proposed by Stock and Watson (1989) has been widely used in that it can express the behavior of a large set of variables using a small number of factors. This model has been especially useful in macro economics and financial econometric; it can be used to extract market portfolio and leading indicators from actual macro and market data. As far as I know, most of research has addressed the estimation of the model, and relatively less attention has been paid to hypothesis testing. The problem of deciding the number of factor was considered only by Bai and Ng (2002) using an information criterion in panel analysis of large cross-sections and large time dimensions. Unlike theirs, this paper considers this problem in the framework of the bivariate time series analysis and proposes the Lagrange multiplier test for the hypothesis that two variables have a single common dynamic factor; the result can be generalized to the case of more than two variables. The Lagrange multiplier test, which requires only the estimation under the null, is useful, because the Wald and likelihood ratio tests require estimation of the large number of factors under the alternative hypothesis. See Stock and Watson (2005) for more recent development of dynamic factor models.

2. MODEL

We now define the following bivariate dynamic factor model:

$$\begin{aligned} y_{1t} &= h_{1t} + v_{1t}, \\ y_{2t} &= h_{2t} + v_{2t}, \quad t = 1, ..., T, \end{aligned} \tag{1}$$

where

$$\begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} \sim NID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \right). \tag{2}$$

This model is an exact dynamic factor model; it has uncorrelated disturbances. The observation density is expressed as

$$\begin{aligned} & f(y_{1t}, y_{2t} \mid h_{1t}, h_{2t}) \\ &= (2\pi\sigma_1^2)^{-1/2} \exp(-(y_{1t} - h_{1t})^2 / (2\sigma_1^2)) \\ & \quad (2\pi\sigma_2^2)^{-1/2} \exp(-(y_{2t} - h_{2t})^2 / (2\sigma_2^2)), \end{aligned} \tag{3}$$

We assume that the unobservable state variables, h_{1t} and h_{2t} , are generated by the following transition equations:

$$\begin{aligned} h_{1t} &= b_1 h_{1,t-1} + \omega_1 u_{1t}, \\ h_{2t} &= b_2 h_{2,t-1} + \omega_2 u_{1t} + \rho u_{2t}, \\ & |b_1| < 1, \ |b_2| < 1, \end{aligned} \tag{4}$$

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \sim NID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \tag{5}$$

The likelihood function of $y_t = (y_{1t}, \ y_{2t})'$ is expressed as

$$\begin{aligned} & f(y_1, ..., y_T) \\ &= \int f(y_T \mid h_T) f(h_T \mid h_{T-1}) \cdots f(y_1 \mid h_1) \\ & \quad f(h_1 \mid h_0) dh_T \cdots dh_1 \end{aligned} \tag{6}$$

by integrating out the latent variable, where

$$\begin{aligned} & h_t = \begin{pmatrix} h_{1t} \\ h_{2t} \end{pmatrix}, \ y_t = \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}, \\ & f(y_t \mid h_t) \equiv f(y_{1t} \mid h_{1t}) f(y_{2t} \mid h_{2t}), \\ & f(h_t \mid h_{t-1}) \\ & \equiv f(h_{1t} \mid h_{1,t-1}) f(h_{2t} \mid h_{1t}, h_{1,t-1}, h_{2,t-1}) \end{aligned} \tag{7}$$

3. TEST STATISTIC

The paper proposes a Lagrange multiplier test for the hypothesis that the two volatilities are proportional $h_{1t} \propto h_{2t}$, which is expressed by the equalities

$$b_1 = b_2, \ \rho = 0. \tag{8}$$

Then, we have only to evaluate the score functions with respect to $b_1 - b_2$ and ρ^2 under the null hypothesis. Note that, although the proportionality is not satisfied exactly under the null on account of the effect of initial values h_{10}, h_{20}, the deviation disappears sufficiently quickly, if $|b_1| < 1, \ |b_2| < 1$. We can also cancel the initial value effect by assuming setting $h_{10} = h_{20}$.

For the sake of notational simplicity, we denote that

$$\begin{aligned} p_{1t} &\equiv f(h_{1t} \mid h_{1,t-1}), \\ p_{2,t} &\equiv f(h_{2t} \mid h_{1t}, h_{1,t-1}, h_{2,t-1}), \\ q_{1t} &\equiv f(y_{1t} \mid h_{1t}), \\ q_{2t} &\equiv f(y_{2t} \mid h_{2t}) \end{aligned} \tag{9}$$

The comparison methods presented here build on recent developments in the land-use modelling and geographical information literature (Hagen 2003, Power *et al.* 2001). They have been adapted to work with the interval/ratio type data that is most commonly produced by hydrological models. These methods are relatively easy to implement and provide useful alternatives to the current methods used in hydrology for comparing spatial fields. The measures produced can be interpreted in a familiar way for hydrologists.

3.1. SCORE WITH RESPECT TO ρ^2

The first derivative of the likelihood function with respect to ρ^2 is expressed as

$$(\partial/\partial\rho^2)\, f(y_1,...,y_T) = K_T + ... + K_1, \quad (10)$$

where

$$K_t = \int\cdots\int \tilde{f}_T \tfrac{1}{2} p_{2t}{}^{-1}(\partial^2 p_{2t}/\partial h_{2t}{}^2)dh \quad (11)$$

$$, \quad \tilde{f}_T = p_{11}p_{21}q_{11}q_{21}\cdots p_{1T}p_{2T}q_{1T}q_{2T}, \quad (12)$$

since we have that

$$\begin{aligned}&(\partial/\partial\rho^2)\, p_{2t}\\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi\rho^2}}\\ &\exp\left(-\frac{(h_{2t}-b_2h_{2,t-1}-\omega_2(h_{1t}-b_1h_{1,t-1})/\omega_1)^2}{2\rho^2}\right)\\ &\left(\frac{(h_{2t}-b_2h_{2,t-1}-\omega_2(h_{1t}-b_1h_{1,t-1})/\omega_1)^2}{\rho^4}-\frac{1}{\rho^2}\right)\\ &= \frac{1}{2}(\partial/\partial h_{2t})^2\, p_{2t}\end{aligned} \quad (13)$$

Then, in integrating out $h_{2T},...,h_{21}$ under the null, we apply the derivative formula of integral of degenerate densities given in Appendix and substitute

$$h_{2t} \equiv b_2h_{2,t-1} + \omega_2(h_{1t} - b_1h_{1,t-1})/\omega_1, \quad (14)$$

which is equation (5) with $\rho^2 = 0$. Noting that we have h_{2t} only in q_{2t} and $p_{2,t+1}$ we have that

$$\begin{aligned}K_T &= \int q_{2T}{}^{-1}(\tfrac{1}{2}\partial^2 q_{2T}/\partial h_{2t}{}^2)\,\tilde{f}d\mathbf{h}\\ K_t &= \int \tfrac{1}{2}(p_{2,t+1}q_{2t})^{-1}\frac{\partial^2(p_{2,t+1}q_{2t})}{\partial h_{2t}{}^2}\,\tilde{f}d\mathbf{h}, \quad (15)\\ t &< T\end{aligned}$$

using the formula of integration by parts. For t=T, we have only to evaluate

$$\begin{aligned}(\partial/\partial h_{2t})q_{2,t} &= A_t q_{2,t},\\ \frac{1}{2}(\partial/\partial h_{2t})^2 q_{2,t} &= B_t q_{2,t}\end{aligned} \quad (16)$$

where

$$\begin{aligned}A_t &\equiv (y_{2t}-h_{2t})/\sigma_2{}^2,\\ B_t &\equiv \frac{1}{2}\left[(y_{2t}-h_{2t})^2/\sigma_2^4 - 1/\sigma_2^2\right]\end{aligned} \quad (17)$$

For $t<T$, we have that:

$$\begin{aligned}&\tfrac{1}{2}(\partial/\partial h_{2,t})^2\left[p_{2,t+1}q_{2t}\right]\\ &= \tfrac{1}{2}q_{2t}(\partial^2 p_{2,t+1}/\partial h_{2,t}{}^2)\\ &+ \tfrac{1}{2}p_{2,t+1}(\partial^2 q_{2,t}/\partial h_{2,t}{}^2)\\ &+ (\partial p_{2,t+1}/\partial h_{2,t})(\partial q_{2t}/\partial h_{2,t})\end{aligned} \quad (18)$$

Then, we have that

$$K_T = E_T,\ K_t = H_t + E_t + G_t,\quad t<T, \quad (19)$$

where

$$\begin{aligned}H_t &= \frac{1}{2}\int p_{2,t+1}{}^{-1}(\partial^2 p_{2,t+1}/\partial h_{2,t}{}^2)\,\tilde{f}dh,\\ E_t &= \frac{1}{2}\int q_{2,t}{}^{-1}(\partial^2 q_{2,t}/\partial h_{2,t}{}^2)\,\tilde{f}dh \qquad ,\\ G_t &= \int p_{2,t+1}{}^{-1}\frac{\partial p_{2,t+1}}{\partial h_{2,t}}q_{2,t}{}^{-1}\frac{\partial q_{2,t}}{\partial h_{2,t}}\,\tilde{f}dh,\end{aligned} \quad (20)$$

We see that

$$\begin{aligned}&H_t\\ &= \frac{1}{2}(-b)^2\int p_{2,t+1}{}^{-1}\frac{\partial^2 p_{2,t+1}}{\partial h_{2,t+1}{}^2}/\ \tilde{f}dh\\ &= b^2K_{t+1}\end{aligned} \quad (21)$$

since, under the null of $b_1 = b_2 \equiv b$, we have that

$$(\partial/\partial h_{2,t})^2 p_{2,t+1} = (-b)^2(\partial/\partial h_{2,t+1})^2 p_{2,t+1}.$$

We also have that

$$E_t = \int B_t\,\tilde{f}dh,$$

$$G_t = \int p_{2,t+1}{}^{-1} q_{2,t}{}^{-1} \frac{\partial p_{2,t+1}}{\partial h_{2,t}} \frac{\partial q_{2,t}}{\partial h_{2,t}} \tilde{f} dh$$
$$= (-b) \int A_t p_{2,t+1}{}^{-1} (\partial p_{2,t+1} / \partial h_{2,t+1}) \tilde{f} dh$$
$$= b \int A_t [q_{2,t+1} p_{2,t+2}]^{-1} \frac{\partial q_{2,t+1} p_{2,t+2}}{\partial h_{2,t+1}} \tilde{f} dh$$
$$= b \int A_t [q_{2,t+1}{}^{-1} \frac{\partial q_{2,t+1}}{\partial h_{2,t+1}} + p_{2,t+2}{}^{-1} \frac{\partial p_{2,t+2}}{\partial h_{2,t+1}}] \tilde{f} dh$$
$$= b \int \tilde{f} A_t [A_{t+1} + (-b) \frac{\partial p_{2,t+2}}{\partial h_{2,t+1}} p_{2,t+2}{}^{-1}] dh$$
$$= b \int A_{t+1} A_t \tilde{f} dh + b^2 \int A_t \frac{\partial p_{2,t+2}}{\partial h_{2,t+1}} p_{2,t+2}{}^{-1} \tilde{f} dh$$
$$= b J_{t,t+1} + b^2 J_{t,t+2} + \ldots + b^{T-t} J_{t,T}$$

where

$$J_{t,s} = \int A_t A_s \tilde{f} dh$$

since we have that

$$\begin{aligned} &\int q_{2,t+1} p_{2,t+2} (\partial p_{2,t+1} / \partial h_{2,t+1}) dh_{2,t+1} \\ &= -\int p_{2,t+1} \partial \left(q_{2,t+1} p_{2,t+2} \right) / \partial h_{2,t+1} dh_{2,t+1} \end{aligned} \qquad (22)$$

from the derivative formula of the integral of degenerate density function.Thus, we have

$$K_T = E_T$$
$$K_{T-1} = b^2 K_T + E_{T-1} + G_{T-1} = b^2 E_T + E_{T-1} + G_{T-1}$$

$$\begin{aligned} K_{T-2} &= b^2 K_{T-1} + E_{T-2} + G_{T-2} \\ &= b^2 (b^2 E_T + E_{T-1} + G_{T-1}) + E_{T-2} + G_{T-2} \end{aligned} \qquad (23)$$
$$K_t = b^{2(T-t)} L_T + b^{2(T-t-1)} L_{T-1} + \ldots + L_t ,$$

where

$$L_t = \begin{cases} E_t + G_t & t < T \\ E_T & t = T \end{cases} \qquad (24)$$

Thus, we can evaluate the test statistic using the conditional covariance of the dynamic factor .

We have that

$$\frac{\partial \log f(y_1, ..., y_T)}{\partial \rho^2} \qquad (25)$$
$$= E[k_1 \mid \mathbf{y}] + \ldots + E[k_T \mid \mathbf{y}]$$

where

$$k_t = b^{2(T-t)} l_T + b^{2(T-t-1)} l_{T-1} + \ldots + l_t$$
$$l_t = \begin{cases} B_t + g_t & t < T \\ B_T & t = T \end{cases}$$
$$g_t = b A_{t+1} A_t + b^2 A_{t+2} A_t + \ldots + b^{T-t} A_T A_t$$

3.2. Score with respect to b_2

We next obtain the first derivative of the log likelihood function with respect to b_2 evaluated at $b_2 = b_1$ and $\rho^2 = 0$ .Under the hypothesis that $\rho^2 = 0$ the transition equation is

$$\begin{aligned} &f(h_{1t} \mid h_{1,t-1}) \\ &= (2\pi\omega_1{}^2)^{-1/2} \exp(-(h_{1t} - b_1 h_{1,t-1})^2 / (2\omega_1{}^2)), \\ &h_{2t} \equiv b_2 h_{2,t-1} + \omega_2 (h_{1t} - b_1 h_{1,t-1}) / \omega_1 \end{aligned} \qquad (26)$$

We have b_2 only in h_{2t} , hence only in $q_{2t} \equiv f(y_{2t} \mid h_{2t})$ Then, we have that

$$\begin{aligned} &\frac{\partial g(y_1, ..., y_T)}{\partial b_2} \\ &= \int p_{1,T} q_{1,T} \frac{\partial q_{2,T}}{\partial h_{2,T}} \frac{\partial h_{2,T}}{d b_2} p_{1,1} q_{1,1} q_{2,1} dh_1 \qquad (27) \\ &+ \ldots + \int p_{1,T} q_{1,T} q_{2,T} \cdots p_{1,1} q_{1,1} \frac{\partial q_{2,1}}{\partial h_{2,1}} \frac{\partial h_{2,1}}{d b_2} dh_1 \end{aligned}$$

where

$$\begin{aligned} &g(y_1, ..., y_T) = f(y_1, ..., y_T) \mid_{\rho=0} \\ &= \int p_{1,T} q_{1,T} q_{2,T} \cdots p_{1,1} q_{1,1} q_{2,1} dh_1 \quad ,(28) \\ &dh_1 = dh_{1,T} \cdots dh_{1,1} \\ &h_{2t} \equiv b_2 h_{2,t-1} + \omega_2 (h_{1t} - b_1 h_{1,t-1}) / \omega_1 \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\partial q_{2t}}{\partial b_2} \equiv \frac{\partial f(y_{2t} \mid h_{2t})}{\partial h_{2t}} \frac{dh_{2,t}}{db_2} \\ &= (2\pi\sigma_2^2)^{-1/2} \exp(-(y_{2t} - h_{2t})^2 / \sigma_2^2) \qquad (29) \\ &\quad n_t (y_{2t} - h_{2t}) / \sigma_2^2 \\ &= q_{2t} n_t (y_{2t} - h_{2t}) / \sigma_2^2 . \end{aligned}$$

where

$$n_t \equiv \frac{dh_{2,t}}{db_2}.$$

Then we have that

$$\frac{\partial g(y_1,...,y_T)}{\partial b_2}$$
$$= \left(1/\sigma_2^2\right)\int n_T(y_{2T}-h_{2T})\tilde{g}dh_1 \quad (30)$$
$$+...+\left(1/\sigma_2^2\right)\int n_1(y_{21}-h_{21})\tilde{g}dh_1$$

where

$$\tilde{g} = p_{T,1}q_{T,1}q_{T,2}\cdots p_{1,1}q_{1,1}q_{1,2},$$

We can easily see that

$$n_t \equiv \frac{dh_{2,t}}{db_2} = h_{2,t-1} + b_2\frac{dh_{2,t-1}}{db_2}$$
$$= h_{2,t-1} + b_2\left(h_{2,t-2} + b_2\frac{dh_{2,t-2}}{db_2}\right) \quad (31)$$
$$= h_{2,t-1} + b_2 h_{2,t-2} + b_2{}^2 h_{2,t-3} + ...$$

Then we have that

$$\frac{\partial \log f(\mathbf{y})}{\partial b_2}$$
$$= \int\left[n_T A_T + ... + n_1 A_1\right] g(\mathbf{h}_1 \mid \mathbf{y})d\mathbf{h}_1 \quad (32)$$
$$= E[n_T A_T \mid \mathbf{y}] + ... + E[n_1 A_1 \mid \mathbf{y}]$$

where the conditional density $g(\mathbf{h}_1 \mid \mathbf{y})$ is defined by

$$g(\mathbf{h}_1 \mid \mathbf{y}) \equiv p_{1,T}q_{1,T}q_{2,T}\cdots p_{1,1}q_{1,1}q_{2,1}/g(\mathbf{y})$$

3.3. Lagrange multiplier test statistic

We have obtained the score function with respect to ρ^2 and b_2. We check the hypothesis using the fact that the expected value of the score functions are zero under the null hypothesis, so that the large deviation of the score function from zero is an evidence that the null hypothesis is not satisfied in reality. The LM test statistic is define by

$$LM = s\Sigma^{-1}s' \quad (33)$$

where

$$s = (\partial \log f / \partial \log \rho^2, \partial \log f / \partial \log b_2)'$$
$$\Sigma = \mathrm{var}(s)$$

Note that we calculate the asymptotic variance taking the estimation error of the estimated parameters $\sigma_1^2, \sigma_2^2, \omega_1^2, \omega_2^2, b_1$. It easy to see that Σ can be calculated by the upper-left 2×2 matrix of the Fisher information of $(\rho^2, b_2, \sigma_1^2, \sigma_2^2, \omega_1^2, \omega_2^2, b_1)'$. It is easy to obtain the Fisher matrix using the BHHH algorithm. (Berndt et al. 1974). See Hamilton (1996) for detailed derivation of the LM test statistic.

4. REFERENCES

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APPENDIX

In the Appendix we obtain the formula for the conditional expected value and covariance of the hidden dynamic volatility given the observed series. The notation employed in this appendix differs from that used in the main text.

Let us denote

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ \vdots \\ y_{1T} \\ y_{21} \\ \vdots \\ y_{2T} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \omega_1 \mathbf{I}_T \\ \omega_2 \mathbf{I}_T \end{pmatrix}, \qquad \text{(A.1)}$$

$$\Omega = \begin{pmatrix} \sigma_1^2 \mathbf{I}_T & \mathbf{0} \\ \mathbf{0} & \sigma_2^2 \mathbf{I}_T \end{pmatrix}, \ \mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_T \end{pmatrix}, \text{(A.2)}$$

$$\mathbf{B} = \begin{pmatrix} \sqrt{1-b^2} & 0 & 0 & 0 \\ -b & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & -b & 1 \end{pmatrix} \text{(A.3)}$$

where $\mathbf{h}$ is the hidden dynamic factor with unit innovation variance and b is the common regression coefficient of the dynamic factor.

The measurement density is expressed as

$$\begin{aligned} &f(\mathbf{y} \mid \mathbf{h}) \\ &= (2\pi)^{-T} \,|\Omega|^{-1/2} \qquad \text{(A.4)} \\ &\exp[-(1/2)(\mathbf{y} - \mathbf{Ah})'\Omega^{-1}(\mathbf{y} - \mathbf{Ah})], \end{aligned}$$

and the transition density is expressed as

$$\begin{aligned} &f(\mathbf{h}) \\ &= (2\pi)^{-T/2} \,|B'B|^{1/2} \qquad \text{(A.5)} \\ &\exp[-(1/2)\mathbf{h}'B'B\mathbf{h}], \end{aligned}$$

Then the conditional density of $\mathbf{h}$ given $\mathbf{y}$ is expressed as

$$\begin{aligned} &f(\mathbf{h} \mid \mathbf{y}) \\ &= (2\pi)^{-T} \,|\mathbf{M}|^{1/2} \qquad \text{(A.6)} \\ &\exp(-(1/2)(\mathbf{h} - c)'\mathbf{M}(\mathbf{h} - c)) \end{aligned}$$

where

$$\mathrm{M} = \mathbf{A}'\Omega^{-1}\mathbf{A} + \mathrm{B'B}, \ \ c = \mathbf{A}'\Omega^{-1}\mathbf{y} \qquad \text{(A.7)}$$

Then the variance covariance matrix of $\mathbf{h}$ given $\mathbf{y}$ is given by M^{-1}.