Construction of the Utility Function Using a Non-linear Best Fit Optimisation Approach

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EXTENDED ABSTRACT

The 1994 water reforms of the Council of Australian Governments (COAG) were the very significant changes in Australia's water resource management practice. The COAG reforms were an integrated package of changes aimed at improving both the efficiency of water use and the condition of Australia's waterways. The major change associated to these reforms was the separation of land and water rights of the owners and the ability to trade these rights separately, subject to ecological, social and financial constraints.

It is hoped that these water market reforms will bring the economics of the agricultural to the situation where all parties are better off: farmers can sell their water rights for their real cost and the water allocation in general shall become more efficient. However, some negative impacts are possible. For instance, will the shift of water rights from the region of lower return to the regions with more profitable industries create the situation when the first region has not enough funds to support the regional water supply infrastructure causing a series of unfortunate social and economic consequences? If water rights are transferred from the predominantly grazing land and pastures of northern Victoria (the Goulburn catchment) to the horticultural region of Sunrasia and the wine producing regions of South Australia, the first region would not be able to maintain the system of reservoirs and water carrier channels impacting negatively on tourism, regional income and employment opportunities in the region. For better prediction of consequences of these reforms, economic modelling is necessary. The computable (CGE) is one of the most powerful tools allowing researchers to deal with this problem.

In the household consumption formulations of CGE modelling a utility function is usually selected *a priori* without taking into consideration the detailed structure of preferences for goods and services. The most commonly used functional forms are the Cobb-Douglas and CES utility functions. Only relative few parameters of these functions are estimated using the real data on consumer preferences, raising questions as to their representativeness. This paper outlines a method via which a purpose built utility function is derived based on real consumer demand data. The method best fits a utility from the class studied by (Afriat, 1967) and is based on revealed preference theory. Such methods can only work exactly if the Generalised Axiom of Revealed Preference (GARP) holds (Fostel, 2004). Consequently a non-liner best fit optimisation algorithm is devised to find the minimum residuals that allow the GARP to hold and hence an Afriat like utility to be fitted. All CGE algorithms are based on a best estimate of the current state of the economy provided by the input-output table. These values maximise the underlying (real) utility subject to the budget constraint. Thus we impose additional constraints to ensure the equilibrium of the fitted utility exists and coincides with the entries in the input-output tables. Thus we arrive at a mathematical program with equilibrium constraints (MPEC), the equilibrium constraints being those associated with the implicit utility maximization problem, tied to the input-output tables estimates. It is well known that MPECs are more difficult to solve than a standard nonlinear optimization problems, often requiring purpose built solvers. In this paper we utilise a mathematical method that allows us to formulate and solve our MPEC as a standard nonlinear programming problem for small data sets.

The goal of this research is to provide a technique to allow researchers to more accurately estimate the cross-elasticities of all commodities included in the modelling. Its intended that the derived utility function will be used to update the current regional CGE model with a view to achieve more accurate realistic predictions.

1 INTRODUCTION

1 Background

The traditional approach to the agricultural market modelling is based on the LP optimisation technique, where the revenue of the regional economy or individual irrigator is maximised subject to the availability of a set of limited resources (land, water, capital, labour etc.) Several economic models have been developed in order to predict volumes and prices of water traded. An overview of such modelling can be found in (Schreider et al. 2005); (Weinmann et al., 2005). The most commonly used model the Australian water market were suggested in works of economists from the Victorian Department of Primary Industry (Eigenraam, 1999) and the Australian Bureau of Resource Economics (McClintock et al.,2000).

The major limitation of the linear optimisation models is that they treat the economic process from the partial equilibrium point of view where the prices for the most part of commodities are given exogenously. The application of computable general equilibrium models (CGE) is a step ahead compared to the partial equilibrium paradigm. CGE models are widely used in Australia (see for instance, Dixon et al., 1982) and world-wide (Ginsburgh and Keyser, 1997). Derivation of the demand functions for each commodity and a primary factor is a central part of any computable general equilibrium (CGE) modelling work.

Traditionally, the formulation of the consumer problem for the CGE model postulates that the explicit form of the utility function is given a priori as some generalisations of the Cobb-Douglas utility functions, demonstrating the constant return to scale, constant elasticity of substitution and constant ratio of elasticities of substitution.

2 Rationales

In this work we apply techniques from generalized convexity, monotonicity and optimisation to the estimation of the utility functions that are used in the CGE modelling of water market and in the agricultural economics. Such estimation techniques will have significant value when applied to recently developed mathematical models of the agricultural economy of northern Victoria. Apart from the pure academic value of the results obtained we expect to improve on the classical CGE economic modelling used to explain and predict water market transactions. Another important rationale of the implemented work is that it allows researchers and water authorities to better understand other socio-economical processes which cannot be described within the classical CGE paradigm. The results obtained form this research project could be used for natural resource management purposes by relevant state authorities for different economic regions, for instance in the regions of irrigated agriculture as the Goulburn catchment, and all around Australia.

3 First pass approach

A central part of any computable general equilibrium (CGE) modelling work is determination of a consumer demand function. This function should be derived for each commodity and primary factor constituting the modelling system. This problem is usually solved as a Lagrangian formulation when a utility function is maximised for a given budget constraint. The maximization of the utility function subject to budget constraints will remain the primary framework but will now provide the upper level part of a bi-level optimization problem. The lower level corresponding a best fit for the utility functions subject to available revealed preference data. This later problem can be treated as a member of the family of Mathematical Programming with Equilibrium Constraints (MPECs).

The Lagrangian technique will not be available since the lower level problem (the best fit utility problem) will only give the utility function indirectly as the solution of an optimization problem. The fitted utility being of the Afriat class is piecewise affine giving rise to a linear program utility maximization problem in the upper level. When we try and fit a concave function to a set of data we need the imposition of additional constraints to select from the equivalence class of utility functions "consistent" with the data. Also when a non-market variable exists it is useful to first fit the indirect utility to the data and infer the direct utility subsequently. Such considerations lead one to consider maximal and minimal representatives of this equivalence class.

This paper reports the first stage of the project where we use artificially simulated bundles of commodities taken from a known utility in order to test our best fit approximation algorithm. Two utility functions, Cobb-Douglas and Constant Elasticity of substitution (CES) were selected as these are commonly used in CGE modelling. The Cobb-Douglas utility is defined as $U(X_i, ..., X_n) = \prod_{i=1}^{n} X_i^{\alpha_i}$, where $\sum_i \alpha_i = 1$. This function represents the demand of commodities with respect to commodity costs and household income. Where α represents the commodity share of good i in the total household expenditure. A larger value of α implies that the commodity holds a greater sales share. Commodity demand of i will decrease with cost increase of commodity i or decrease in household income. Price elasticity of demand is unitary, therefore a 10% increase in price will lead to a 10% decrease in demand. Cross-price elasticity for this case is zero, hence the demand of one commodity is not dependent on the other. The CES utility function is defined as $U(X_i, ..., X_n) = \left(\sum_{i=1}^{n} \alpha_i X_i{}^{-\rho}\right)^{-\frac{1}{\rho}}$, where $\rho = \frac{1-s}{s}$ $(s > 0, s \neq 1)$. Commodities are considered to be: 1). Unitary Elastic: If as $s \to 1$the CES function behaves like the Cobb-Douglas function. 2). Perfect Complements: As $s \to 0$ the CES approaches the Leontief function where commodities are consumed jointly to satisfy the consumer. Hence price elasticity of demand is inelastic as a consumer will not give up more of one commodity for the other. 3). Perfect Substitutes: As $s \to \infty$ both goods can be substituted whilst still achieving maximum utility. This displays price elastic demand. The algorithm used in the approximation is detailed in Section 2 of this paper.

2 METHOD

A utility function $u : \Re^n_+ \to \Re$ is often used to reflects the preference structure with respect to possible consumption. It is well known that under minimal assumptions we can define a consumer preference $y\mathcal{R}x$ via a utility using $y \in S_u(x) := \{z \in K \mid u(z) \geq u(x)\}$. It is natural to assume u is non-decreasing on it domain $\Re^n_+$ of positive commodity bundles and nonsatiated (i.e. "no flats"). A preference $\mathcal{R}$ induces a demand relation $D_\mathcal{R}$ in that $(x, x^*) \in D_\mathcal{R}$ if and only if we are within budget $x \in BG(x^*) := \{y \geq 0 \mid \langle y, x^* \rangle := y_1 x_1^* + \cdots + y_n x_n^* \leq 1\}$ and $x\mathcal{R}y$ for all $y \in BG(x^*)$ (we have rescaled the unit of money so that the budget $w = 1$). This amounts to the solution of the following optimization problem (parameterised by x^*) $v(x^*) := \max\{u(x) \mid \langle y, x^* \rangle \leq 1\}$, which defines the so called indirect utility v. Under mild assumptions we may recover the direct utility via the duality formula $u(x) = \inf\{v(x^*) \mid \langle x, x^* \rangle \leq 1\}$.

In consumer preference theory we have only access to finite sample of such pairs $\mathcal{X} := \{(x_i, x_i^*) \in D_\mathcal{R},\ i \in I\}$ where $I := \{1, \ldots, m\}$. We say that x is a revealed preference to y when $(x, x^*) \in D_\mathcal{R}$ and $\langle x^*, x - y \rangle \geq 0$n denoting this by $x \succeq_{D_\mathcal{R}} y$. That is, y was in budget as $\langle x^*, x \rangle \geq \langle x^*, y \rangle$ but as $(x, x^*) \in D_\mathcal{R}$ we have x chosen instead of y. Such a *finite expenditure configuration*, $\mathcal{X}$ gives rise to a partial order $\succeq_R$via the transitive closure of $\succeq_{D_\mathcal{R}}$where $x \succeq_R y$ when there exists $x = x_0, x_1, \ldots, x_n = y$ with $x_{i+1} \succeq_{D_\mathcal{R}} x_i$ for all i. Similarly we denote $x \succ_R y$ when $x \succeq_R y$ and there exists i with $x_{i+1} \succ_{D_\mathcal{R}} x_i$ or $\langle x_{i+1}^*, x_{i+1} - x_i \rangle > 0$ for $(x_i, x_i^*) \in D_\mathcal{R}$. The *generalised axiom of revealed preference* (GARP) says that there can not exists a cycle $\{(x_i, x_i^*) \mid i = 0, \ldots, n\}$ (with $x_0 = x_{n+1}$) such that all $\langle x_{i+1}^*, x_{i+1} - x_i \rangle \geq 0$ unless $\langle x_{i+1}^*, x_{i+1} - x_i \rangle = 0$. The GARP is necessary and sufficient for the existence of a preference order $\succeq$ on $\mathcal{X}$ such that $x \succeq y$ whenever $x \succeq_R y$ and $x \succ y$ whenever $x \succ_R y$. That is $\succeq_R$ rationalises $\mathcal{X}$. The fundamental problem of consumer preference theory is the question as to how to fit an unknown utility u given only a *finite expenditure configuration*, $\mathcal{X}$.

This brings us to the classical work of (Afriat, 1967) and (Fostel, 2004). Under the assumption of the GARP there is a feasible solution to the "Afriat" inequalities

$$\phi_j \leq \phi_i + \lambda_i \langle x_i^*, x_j - x_i \rangle \quad \text{for } i, j \in I. \qquad (1)$$

When we have such a solution we may then define a concave utility via

$$\begin{aligned} u^-(x) := \min\{&\phi_1 + \lambda_1 \langle x_i^*, x - x_1 \rangle, \ldots \\ &\ldots, \phi_m + \lambda_m \langle x_m^*, x - x_m \rangle\} \end{aligned}$$

with the properties: 1). $u(x_j) = \phi_i$ for $j \in I$ since by definition and the Afriat inequalities

$$\begin{aligned} u^-(x_j) &= \min\{\phi_1 + \lambda_1 \langle x_i^*, x_j - x_1 \rangle, \ldots \\ &\qquad \ldots, \phi_m + \lambda_m \langle x_m^*, x_j - x_m \rangle\} \\ &= \phi_j + \lambda_j \langle x_j^*, x_j - x_j \rangle = \phi_j. \end{aligned}$$

2). If x is within budget for price x_i^* but is not preferred to the "revealed preference" x_j i.e.

$$\langle x_j^*, x \rangle \leq \langle x_j^*, x_j \rangle (= 1) \quad \Rightarrow \quad u^-(x) \leq u^-(x_j)$$

since $u(x) \leq \phi_j + \lambda_j \langle x_j^*, x - x_j \rangle \leq \phi_j = u^-(x_j)$.

3). Also

$$\begin{aligned} u^-(x) - u^-(x_i) &\leq \phi_i + \lambda_i \langle x_i^*, x - x_i \rangle - \phi_i \\ &= \langle \lambda_i x_i^*, x - x_i \rangle \end{aligned}$$

or equivalently $\lambda_i x_i^* \in \partial(-u^-)(x_i)$.

What happened when there are errors and the GARP does not hold? We assume the error in GARP is due to inaccurate values of $\{x_i\}_{i \in I}$ then we need to introduce errors $\{s_i\}_{i \in I}$ and move x_i to $x_i + s_i$ (but enforce $s_0 = 0$) and consider:

$$\min_{(\phi, \lambda, s)} \sum_{i \in I,\ i \neq 0} |s_i|^2$$

subject to

$$\begin{aligned} \phi_j - \phi_i \leq \lambda_i \left[\langle x_i^*, x_j - x_i \rangle + \langle x_i^*, s_j - s_i \rangle\right] \\ \text{for } i, j \in I \quad \text{and for all } i, \quad \lambda_i \geq 1. \end{aligned} \qquad (2)$$

and placing $u^{-}(x) = \min\{\phi_i + \lambda_i\langle x_i^*, x - x_i - s_i\rangle\}$ for all x.

In practise economist have merged a set of data $\{(x_i, x_i^*)^*\}_{i\in I}$ into a one estimate of the true equilibrium state of the system (x_0, x_0^*). Thus we require (x_0, x_0^*) to be a solution of the optimization problem. Let w denote the budget then

$$\max_x \quad t$$
$$\text{Subject to} \quad u^{-}(x) \geq t \quad \text{and } \langle x, x_0^*\rangle = w$$

This has a Lagrangian

$$L(x, \eta_0, \eta) := t - \eta_0(\langle x, x_0^*\rangle - w)$$
$$-\sum_i \eta_i(t - \phi_i - \lambda_i\langle x_i^*, x - x_i - s_i\rangle)$$

which gives rise to the optimality conditions (replacing $\eta_i \leftarrow \eta_i/\eta_0$ so that $\sum_{i=1}^m \eta_i = 1/\eta_0 > 0$ assuming $\langle x, x_0^*\rangle = w$)

$$x_0^* = \sum_i \eta_i\lambda_i x_i^* \quad \text{with } \eta_i \geq 0$$
$$0 \geq t - \phi_i - \lambda_i\langle x_i^*, x - x_i - s_i\rangle \quad ,$$
$$\text{and} \quad \eta_i(t - \phi_i - \lambda_i\langle x_i^*, x - x_i - s_i\rangle) = 0.$$

Including this into the utility fitting problem (assuming $\langle x_0, x_0^*\rangle = w$) we arrive at the optimization problem

$$\min_{(\phi,\lambda,s,t,\eta)} \sum_{i\in I,\ i\neq 0} |s_i|^2 \qquad \text{(U-MPEC)}$$

subject to

$$\phi_j - \phi_i \leq \lambda_i[\langle x_i^*, x_j - x_i\rangle + \langle x_i^*, s_j - s_i\rangle]$$
$$\text{for } i, j \in I, \quad x_0^* = \sum_i \eta_i\lambda_i x_i^* \quad \text{with}$$
$$\kappa_i = \lambda_i\langle x_i^*, x_0 - x_i - s_i\rangle - t + \phi_i,$$
$$\lambda_i \geq 1, \quad \eta_i \geq 0 \quad \text{and} \quad \kappa_i \geq 0$$
$$\text{and} \quad \eta_i\kappa_i = 0 \quad \text{for all } i$$

which is a mathematical program with equilibrium constraints (MPEC). One may easily change the norm used in the objective substituting it with the one norm (the sum of modulus) or the infinity norm (the maximum of modulus) i.e.

$$\sum_{i\in I,\ i\neq 0} |s_i| \ \text{ or } \max_{i=1,\ldots,m} |s_i|.$$

This may improve performance in the presence of outliers.

Usually such problem need purpose built solvers but by relaxing the equilibrium constraint $\eta_i\kappa_i = 0$ to $\eta_i\kappa_i \leq 0$ in conjunction with $\eta_i \geq 0$ and $\kappa_i \geq 0$ we obtain an equivalent formulation to which the standard sequential quadratic programming problem is applicable when sample size m are less than 25. If one tries fits the indirect utility $-v$ to the inverted data $\mathcal{X}^T := \{(x_i^*, x_i) \mid (x_i, x_i^*) \in \mathcal{X}\}$ one obtains v^+ and by association $u^+(x) := (v^+)^*(x) = \inf_{x^*}\{\langle x, x^*\rangle - v^+(x^*)\}$ which can be shown to be the largest concave utility consistent with $\mathcal{X}$ while u^- is the smallest.

3 RESULTS

1 Cobb-Douglas Utility

Random price data was generated for 20 samples of commodity bundles of size 2. The demand data was obtained for both commodities by maximising the Cobb-Douglas utility function subject to the pricing constraint. The first pricing samples give X_1 and X_2 as 0.8 and 0.6 respectively where this represents the system at equilibrium. Small randomness in the demand data was added to each commodity demand as to model small "hiccup" in the system or possible errors in the data gathering process. In all simulations we do not perturb the first data value as this represents the true state of the system and this value has been circled in blue in our plots. The parameter α was varied to demonstrate uniform shares, biased shares towards commodity X_1 and strongly biased shares towards commodity X_1.

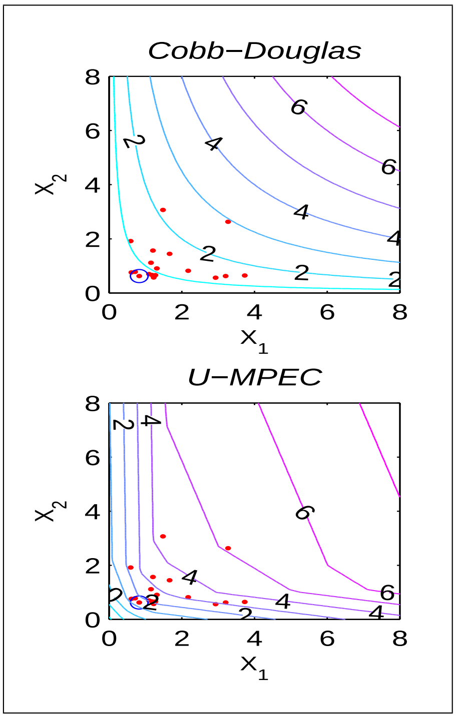

Figure 1. Equal shares of commodities X_1 and X_2 with $\alpha = 0.50$.

The level curves show indifference towards commodity bundles and it can be concluded that the consumers satisfaction will be reached by any choice made along the level curve. A consumer will always choose to maximise utility therefore a commodity bundle that sits on the upmost level curve that intersects the budget constraint will always be chosen. For the 20 samples chosen a combined value of commodities 1 and 2 are consumed to maximise consumer utility based on the pricing bundle.

In Figure 1 a uniform distribution of household shares of commodities 1 and 2 is displayed. The "true" optimal value $(X_1, X_2) = (0.8, 0.6)$ is contained in the blue circle. The U-MPEC utility gives a similar curve to the Cobb-Douglas where the data is more congested. As the data disperses the utility approximation flattens out.

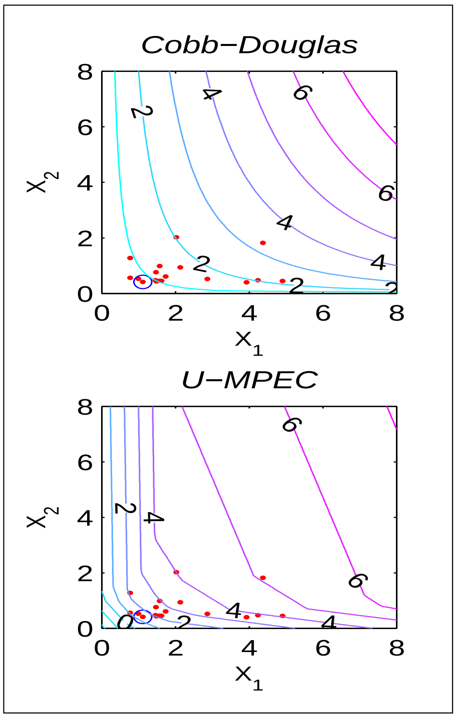

Figure 2. Biased shares of commodity X_1 with $\alpha_1 = 0.67$ and $\alpha_2 = 0.33$.

In Figure 2 demand with a biased towards commodity X_1 has shifted data so that utility is maximised for larger quantities of commodity X_1 with less of commodity X_2 consumed. The "true" optimal value $(X_1, X_2) = (1.1, 0.4)$ is contained in the blue circle. Level curves in both graphs decline more steeply to accommodate the shift in demand. The utility approximation is comparing well with the Cobb-Douglas utility.

In Figure 3 strong biasedness of commodity X_1 has dramatically shifted the data toward larger amounts of X_1 consumed to very little of X_2. The "true" optimal value $(X_1, X_2) = (1.3, 0.25)$ is contained in the blue circle. Again the level curves in both graphs decline more steeply to accommodate the shift in demand with the utility approximation representing the Cobb-Douglas utility very well.

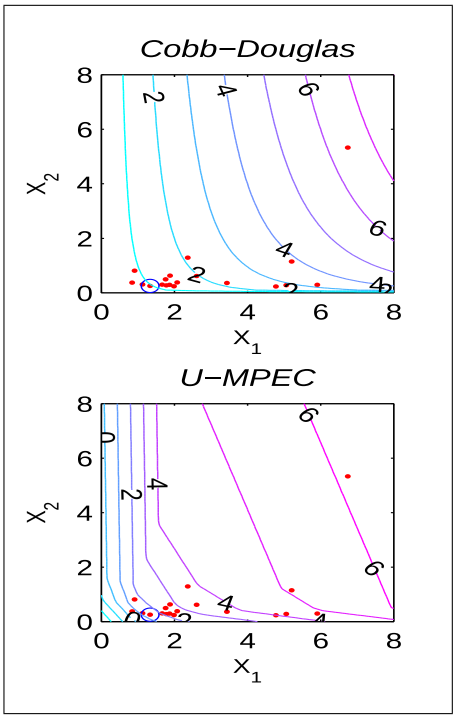

Figure 3. Strongly biased shares of commodity X_1 with $\alpha_1 = 0.80$ and $\alpha_2 = 0.20$.

2 CES Utility

The CES utility function was used to calculate demand based on the same price data used in the Cobb-Douglas utility problem.

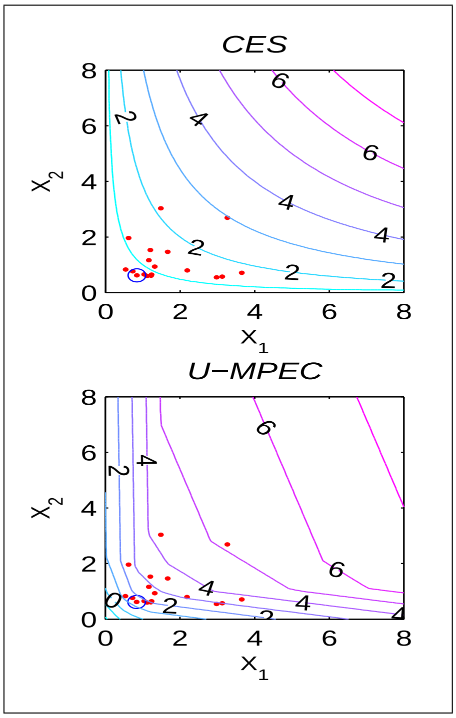

Figure 4. Typical CES curve with $s = 1.1$ and $\alpha = 0.50$.

Typically s is chosen to be 1.1 as it closely represents the Cobb-Douglas utility function with slightly greater price elasticity of demand. Hence a 10% increase in price will lead to a 11% decrease in demand. In Figure 4 the CES utility approaches unitary elasticity as the value of s is close to 1. The "true" optimal value $(X_1, X_2) = (0.8, 0.6)$ is contained in the blue circle.The U-MPEC has compares well with the level curves of the CES utility around the region of the data set but the approximation appears to degrade as we move away from the region within which the simulated data falls.

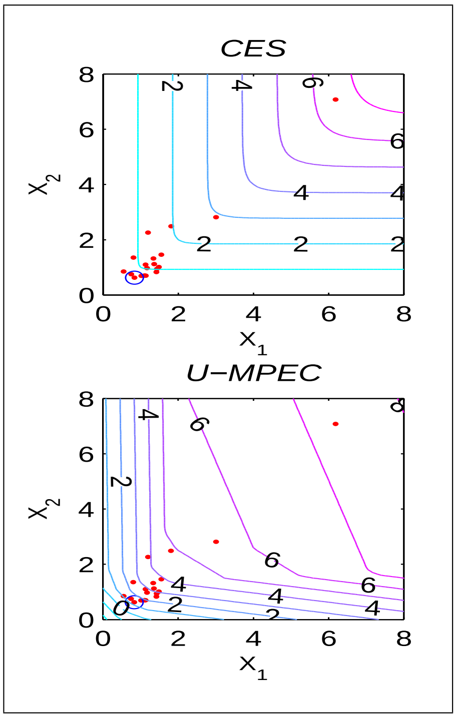

Figure 5. Leontief type curve with $s = 0.1$ and $\alpha = 0.50$.

In Figure 5 the commodities are treated are perfect substitutes hence the level curves are L shaped and data points are bundled together at the base of the curve. The consumer will only increase utility if they can increase both commodities jointly. The U-MPEC has also adjusted the shape of level curves to fit the given data. Noting the outlier in the data has occurred due to small price data being generated for both commodities X_1,X_2 hence allowing the consumer to increase buying power. The "true" optimal value $(X_1, X_2) = (0.8, 0.6)$ is contained in the blue circle. Again the approximation is good close to the cluster of simulated data values but degrades as we move far away from these values.

4 DISCUSSION AND CONCLUSION

Consumers try to maximise their utility (satisfaction) level given a budget constraint. However, generalised utility functions like the Cobb-Douglas and the CES studied in this paper force the demand data to fit the desired curve based on preconceptions of consumer choices. Inconsistences in consumer choices and randomness in commodity selection are not considered. Data is only used from the Input-Output table to calculate share values of commodities. In this case the household share value was calculated as the amount of a commodity used with respect to total household spending.

The U-MPEC formulated has allowed the user to fit the function to given data using the 2 norm without constraining consumer choices to given demand functions. It is unrealistic to assume that all consumers demand follows a Cobb-Douglas unitary price elasticity of demand when many consumer choices are much more complex. In the case of perfect substitutes, consumers are represented as being indifferent to the choice of a commodity provided they are similar. Many examples given in text books describe Coca-Cola and Pepsi as perfect substitutes. Supposedly consumers are indifferent to the choice between the two commodities provided the price is within budget.

The small, artificially generated samples have been used to demonstrated the ability of the U-MPEC utility function to fit two commonly used utility functions. Qualitatively the U-MPEC utility compares surprisingly well to these given utility functions, considering that such a small sample size was used. This is particular true for the Cobb–Douglas utility. For the CES utility the approximates appears to be best within the region that the data is clustered. It is conjectured that the good quality of the U-MPEC approximation is due to the highly constrained nature of the MPEC optimisation problem. Thus small amounts of data appear to be sufficient to replicate some of the general utility functions commonly used although further studies will be need to confirm this beyond reasonable doubt. Further studies in substituting the norm used in the objective function to the 1 or infinity norm will determine whether in fact it may improve performance in the presence of outliers. It would be desirable to develop a statistical theory to estimate errors that result from the propagation of random fluctuations in data values to our utility approximation.

Due to the inability to apply standard optimization algorithms to solve the U-MPEC problem for larger samples, the next step is to develop a purpose built solver for larger data sets. This will be important to further test our approximation method and important in order to apply this method to the regional CGE model. We may then test if this method allows more accurate modelling of consumer demand.

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