Deterministic Geometric Modeling of Natural Complexity

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EXTENDED ABSTRACT

Modeling of nature's complexity is among the outstanding challenges we are faced with in the geophysical sciences. Although many ideas have been proposed throughout the years, the most common approaches, being based on notions related to chance, turn out to be insufficient to study, on an individual basis, the vast variety of patterns seen in nature. To this end, a novel deterministic fractal geometric procedure producing a host of interesting patterns over one, two or three dimensions, as transformations of multifractal distributions via fractal functions, is reviewed in this work. Inspired by the success in representing fluid turbulence via multiplicative cascades yielding multifractal measures, the novel approach represents other geophysical phenomena as a form of a "fractional integration" of such an underlying turbulence process performed using a suitable fractal function that transforms, say turbulence into rainfall.

The ideas are illustrated via various examples over one and higher dimensions that include the modeling of a rainfall time series in Boston, USA and pollution concentration patterns at the Borden site in Canada. It is shown how the geometric procedure results in faithful deterministic representations of actual geophysical patterns that are wholistically defined with substantial compression ratios that often exceed 100:1. It is also explained how the ideas could lead to better understanding of the dynamics of evolving patterns by focusing on how the few parameters of consecutive patterns evolve.

It is argued that finding simplicity at the root of complexity is an important challenge in science for years to come. Undoubtedly, geophysical complexity is very hard to quantify and as such there are, no doubt, other opportunities for improvement via extensions of the notions presented here and others. In regards to the work presented herein, it is recognized that more research is needed in order to solve a non-trivial inverse optimization problem for a given data set. It is also concluded that additional insight needs to be gathered so that the fractal gemetric procedure may be fully understood in terms of commonly defined physical knowledge, such as conservation principles and differential equations.

1. INTRODUCTION

With recent technological advances and the development of sophisticated mathematical techniques, such as those based on fractal geometry, modeling of nature's complexity has attained a new level. Although these ideas have resulted in a new language to describe the intricacies of data sets at hand, oftentimes such tools are insufficient to study, on an individual basis, the incredible variety of patterns available to us, say in geophysical applications.

Since natural sets, such as time series, spatial patterns, and space-time sets, are typically erratic, noisy, intermittent, complex, or in short "random," it has become natural to use stochastic (fractal) theories in order to model them. This has given rise to a variety of approaches that even though yield modeled sets, i.e. realizations that preserve relevant statistical and physical attributes of the records (e.g. autocorrelation function, power spectrum, moments, etc.), such are often unable to capture the specific details and textures found in individual data sets.

Given that stochastic approaches, by definition, can only generate plausible realizations preserving some of the features, but not all of them, one is interested in, and as studies of nonlinear dynamics and deterministic chaos have revealed to us that details indeed matter (e.g. in climate studies; Lorenz 1963), the following questions arise: (1) Could it be possible to find suitable models of individual patterns that capture not only the overall trends and statistical features of the records but also their inherent details? (2) Could such a modeling approach help explain deterministically what otherwise appears to be random, as in deterministic chaos? and (3) Could such ideas, by capturing details, be helpful in studying the underlying dynamics of such sets?

Encouraged by the success in defining certain deterministic fractal sets via iterations of simple maps (e.g. Barnsley 1988), this work reviews a fractal geometric approach aimed at capturing the complexity of natural patterns. As shall be demonstrated herein, the geometric approach produces a vast class of patterns, defined over one, two, and higher dimensions, that resemble those found in a variety of geophysical applications, and that are defined as deterministic derived measures obtained transforming simple multifractal measures via fractal interpolating functions (e.g. Puente 1992).

It is illustrated how this framework leads to interesting data sets, fully characterized in terms of few geometric parameters (i.e. the quantities that define the fractal function and the simple multifractal), that closely resemble geophysical patterns, such as rainfall time series and two- and three-dimensional pollution plumes.

The organization of this paper is as follows. Given first is a review of the mathematical construction. This is followed by a variety of interesting examples that include irregular patterns over one and higher dimensions and applications to geophysical data sets. The article ends with its concluding remarks.

2. A FRACTAL GEOMETRIC APPROACH

The graph G of a fractal interpolating function, shaped as a "wire" from x to y and passing by N+1 points on the plane $\{(x_n, y_n): x_0 < x_1 < ... x_N\}$, is defined as the unique attracting set of N simple affine maps as follows:

$$w_n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix} \quad (1)$$

subject to the conditions:

$$w_n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \quad (2)$$

$$w_n\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad (3)$$

and

$$|d_n| < 1. \quad (4)$$

Equations (2) to (4) ensure that the attractor G exists and that it contains the initial interpolating points. They also allow computing the wire parameters a_n, c_n, e_n, and f_n, in terms of the vertical scalings d_n and the coordinates of the interpolating points, via simple linear equations (e.g. Barnsley 1988; Puente 1992). At the end, a unique, and hence deterministic, set G is found that turns out to have a fractal dimension D between 1 and 2 (e.g. Puente 1994).

In a practical setting, the graph of a fractal wire is obtained sampling the unique attractor dot by dot, starting the process at a point already in G and progressively reiterating the affine maps w_n according to, for example, the outcomes of independent "coin" tosses (e.g. Barnsley 1988). As this process is carried out, it happens that a unique invariant measure is also induced over G that reflects how the attractor is being filled. The existence of such a measure allows computing unique (and fully deterministic) projections over the coordinates x and y (namely dx and dy) that turn out to have irregular shapes as found in applications (e.g. Puente 1996).

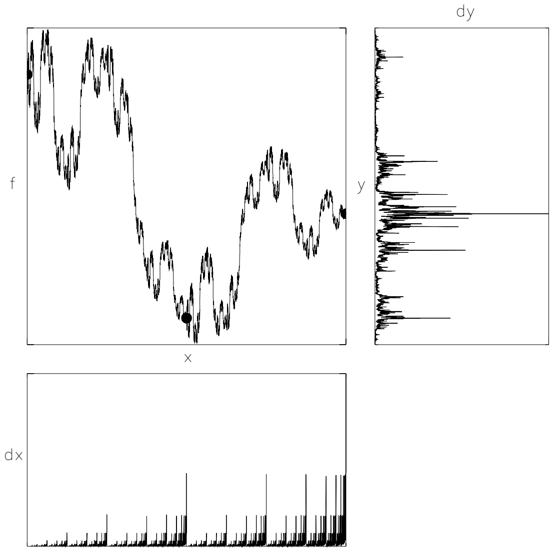

Figure 1. The fractal-multifractal framework. A multifractal measure dx is transformed via a fractal interpolating function f into a derived measure dy. (The scale in x is from 0 to 1, and the one in y is from -0.38 to 0.06. The vertical scales in dx and dy are not given here but both measures are normalized so that they add up to one).

Figure 1 shows an example of these ideas for a fractal function that passes by the three points {(0,0), (1/2,-0.35), (1,-0.2)}, when the scalings of the two affine maps are d_1 = -0.8 and d_2 = -0.6. In addition to the attracting fractal wire f, the figure includes the implied projections dx and dy of the unique measure over the graph of f when the corresponding mappings w_1 and w_2 are iterated according to a 30-70% proportion, using "independent" pseudo-random numbers, starting the process from the mid-point (1/2,-0.35).

Notice how, given the lack of dependence of y on x, i.e. on the first component of the affine mappings (Eq. 1), the implied measure over x is simply a deterministic binomial multifractal measure (Mandelbrot 1989). The measure dy, in turn, being related to dx via the deterministic fractal wire (one that has "noticeable" repetitions), is just the derived measure of dx via the wire f and is, hence, computed looking at all possible heights y and adding the corresponding "probabilities" from "events" that emanate from x (e.g. Puente 1994).

As is seen, the ideas lead to very interesting and random-looking measures dy, which as in the above example resemble, for instance, a rainfall data set as a function of time (e.g. Puente and Obregón 1996). As multifractal measures have been found relevant in studies of turbulence (e.g. Meneveau and Sreenivasan 1987), the projection sets given by these ideas, which turn out to perform a non-trivial fractional integration of a simple parent multifractal measure over x, may be assigned an interpretation as reflections or transformations of turbulence (e.g. Puente et al. 1999). In what follows, it shall be shown how these ideas, and their extensions to higher dimensions, may be employed to represent a variety of complex natural sets over one and higher dimensions.

3. ONE-DIMENSIONAL PATTERNS

To illustrate the wide variety of patterns that may be generated via the fractal-geometric framework, this section includes few examples of one-dimensional sets found from wires that pass by three interpolating points, and that resemble actual geophysical sets.

Figure 2 shows five interesting "rainfall" like patterns made of 4096 points that include, in the middle, the one given in Figure 1. They all share the same parameters, except for their y_1 value of the middle interpolating points, which ranges (from top to bottom) from -0.95 to 0.25 in increments of 0.3.

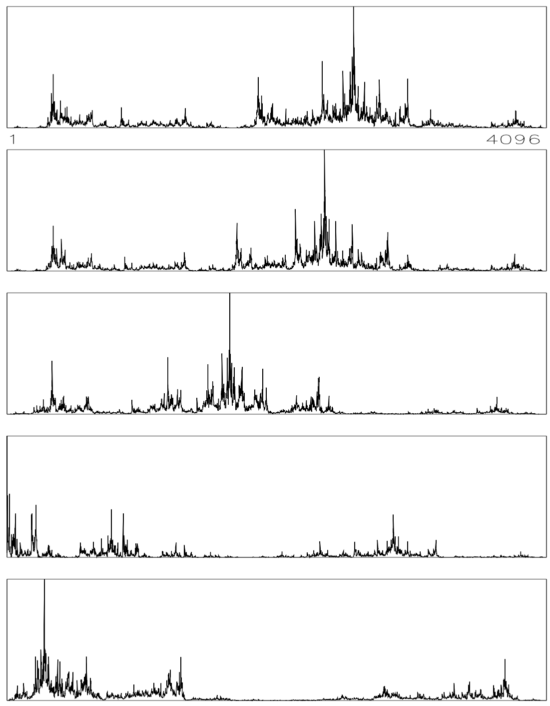

Figure 2. Examples of one-dimensional deterministic measures found varying the interpolating coordinate y_1. All patterns are normalized and have similar vertical scales.

Notice how the major peaks in these sets typically move from right to left and how these images, coming from fractal wires of equal dimensions, share similar intermittencies and textures. It also happens that such sets exhibit similar power-law power spectrum behaviors with spectral exponents in the vicinity of 1.27 and have varying degrees of decay in their autocorrelation functions [Figures not shown], as encountered in geophysical applications and others (Puente 2004).

Figure 3 further illustrates the kinds of one-dimensional sets (made of 4096 points) that may be obtained via the fractal geometric ideas, once again using fractal wires passing by three interpolating points.

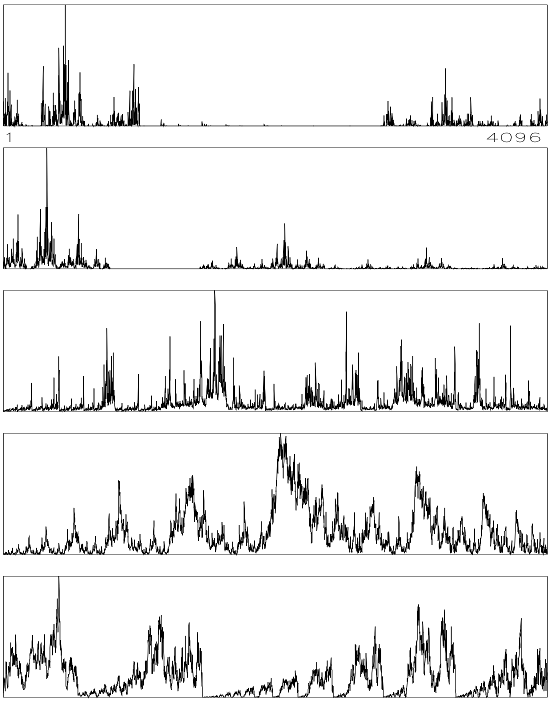

Figure 3. Interesting one-dimensional patterns generated via the fractal geometric approach. All patterns are normalized and have distinct scales.

The two sets at the top of Figure 3 (just like the set in the bottom of Figure 2) contain long periods of close-to-no activity, and hence are useful to represent fully intermittent processes. This is a surprisingly welcomed feature springing out of a single deterministic wire, especially when even the most sophisticated stochastic approaches that currently exist do not seem to be able to capture such transitions from activity to lack of activity (e.g. zeros in rainfall, river flows, etc.).

The middle set in Figure 3 shows a pattern exhibiting a highly irregular and seemingly random structure that nonetheless is fully deterministic. The last two sets in the figure show that the fractal geometric approach can also generate structures that exhibit downward and upward ramps of decay and growth, respectively, as found in applications.

At the end, the fractal geometric approach, using also wires that pass by more than three interpolating points, produces indeed a very vast number of interesting deterministic patterns that not only share the overall shapes found in geophysical sets and others but also maintain the typical autocorrelations and power spectra of natural sets (Puente 2004).

4. SOME GEOPHYSICAL APPLICATIONS

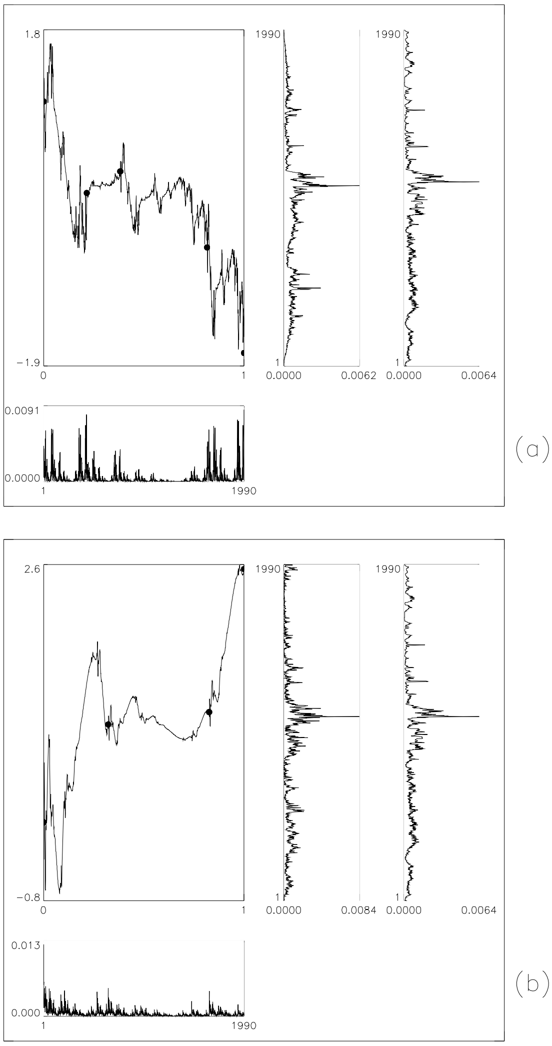

Figure 4. A rainfall storm in Boston (far right) and the construction of derived measures as given by five and four interpolating points, respectively.

The fractal geometric approach has been used in an inverse-problem mode in an attempt to model real data sets, including rainfall (e.g. Puente and

Obregón 1996), turbulence (e.g. Puente and Obregón 1999), width functions of natural channels (e.g. Puente and Sivakumar 2003), and contaminant transport in groundwater (e.g. Puente et al. 2001a, b), among others.

As a way of illustration, Figure 4 shows two alternative representations for a rainfall data set gathered in Boston, USA (made of 1990 data points every 15 seconds) from two fractal wires that pass by five (top) and four interpolating points (bottom).

As may be seen, the deterministic representations via the fractal geometric approach do capture the essential features (i.e. overall shape and detailed texture) of the rainfall event. Clearly, a casual observer would not be able to discriminate between the three “rainfalls” in Figure 4(a) and 4(b). In fact, when statistical and dynamic analyses are performed on such sets, not even an expert would call them different, because they share a host of statistical qualifiers that include moments, multifractal mass exponents, autocorrelation functions, and rainfall intensity histograms, and they can also be classified as coming from low-dimensional chaotic systems of similar dimensions (Puente et al. 2002).

In view of the above, although the actual data set is not fully reproduced by the representations obtained from the fractal geometric approach, the merits of the notions are obvious. Usage of a fractal wire results indeed in parsimonious models of whole data sets, which for the Boston storm give substantial compression ratios of 1990:17 and 1990:12, respectively, or 117:1 and 166:1 for the wire passing by five and four interpolating points, respectively.

Analysis with other data sets (not presented herein) reveals faithful representations via the fractal geometric approach with similar, and sometimes even higher, compression ratios (e.g. Obregón et al. 2002a, b). That sensible approximations may be obtained is by now an established fact, but defining a universal inversion algorithm applicable to all circumstances remains an unresolved problem and a topic of relevant research.

5. HIGHER-DIMENSIONAL PATTERNS

The expressions presented in Equations (1) to (4) may be extended to higher dimensions, so that such generalized mappings produce attracting fractal wires living in three or four dimensions. Such objects may then be used to calculate joint derived measures (over planes and volumes) that turn out to define interesting higher-dimensional patterns.

Figure 5, for example, stemming from a fractal wire defined from x into y, z, and w, shows a complex texture over three dimensions that reflects how the wire is sampled (typically yielding a multifractal dx over x as before, Figure 1) and how such is convolved in a four-dimensional space (x, y, z, w) via the specifics of the wire. As is seen, the resulting set, in the y-z-w space, is reminiscent of a still snap shot of a pollution plume, and so it happens with the corresponding two-dimensional projections over the planes y-w, y-z, and z-w.

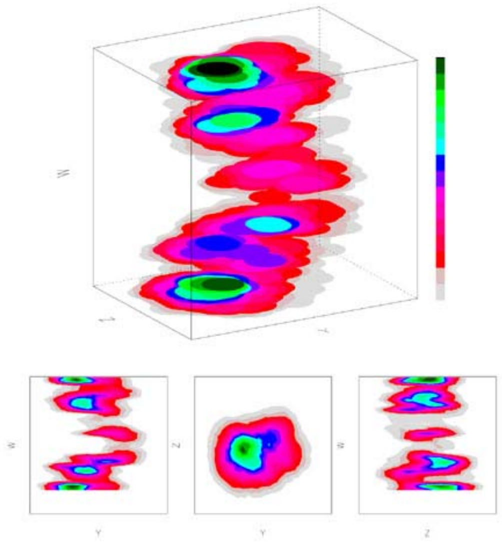

Figure 5. A suitable three-dimensional concentration pattern and its two-dimensional projections as generated via extensions of the fractal-multifractal approach.

The ideas herein have been successfully applied to the modeling of a sequence of pollution snap shots reflecting the evolution of a pollutant. In such a case, at the Borden site aquifer in Canada, faithful representations of vertically averaged (over the plane y-z) concentrations led to the elucidation of simple trends in wire parameters over time that allowed predicting the fate of the pollutant based on the geometry of the plume and (surprisingly) without the need of stochastic partial differential equations (e.g. Puente et al. 2001a, b).

6. CONCLUSIONS

It has been illustrated that the usage of fractal functions and multifractal measures yields a multitude of suitable patterns that resemble those found in geophysical applications, and that such representations may be obtained with substantial compression ratios exceeding 100:1. As it has been explained herein, the geometric procedure turns out to provide a viable alternative to existing procedures based on stochastic methods, one that, in a counter-intuitive fashion, also hints at the possibility of hidden determinism in natural complexity.

It is envisioned, pending a resolution of the required inverse problem, that the fractal geometric approach and its extensions (using non-affine maps or fractal surfaces rather than fractal wires), and other procedures aiming to capture mathematical morphology explicitly, would result in improved understandings of complex natural patterns and their dynamics. It is also envisioned that physical knowledge, as defined via conservation principles and differential equations, may be coupled with the geometric ideas herein.

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