Tests for Jumps in Stochastic Volatility Processes

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EXTENDED ABSTRACT

This paper proposes the Lagrange multiplier (LM) test for the null of the simple stochastic volatility (SV) model without jumps against the alternative of the SV model with jumps in return. It is shown that the LM test statistic does not include the jump probability, which is an unidentified parameter under the null hypothesis, and that this test is free from the Davies problem (1977). The LM test for jumps in volatility is also proposed in the case where jumps in returns and volatility are contemporaneous and correlated under the alternative hypothesis. Dirac's delta function method is used in dealing with the degenerate likelihood function of volatility jumps with infinitely small variance under the null. It is also shown that this test statistic is also free from the unidentified parameter under the null. The LM test statistic for volatility jumps cannot be obtained in the case where jumps in returns and volatility are stochastically independent. The estimation of the stochastic volatility (SV) models with jumps has been an important topic in financial econometrics, because the excess kurtosis that cannot be explained by the simple SV model is often attributed to jumps in returns and volatility.

This paper considers the following four types of stochastic volatility models: Simple SV Model without Jumps (Simple SV), SV Model with Jumps in Returns (SVJ), SV Model with Independent Jumps in Returns and Volatility (SVIJ), SV Model with Correlated Jumps in Returns and Volatility (SVCJ), which are discretized versions of the continuous models of Eraker et al. (2003).

In contrast to the large number of papers on estimation of the jump models, less attention has been paid to testing. The presence of jumps has not been checked by the standard tests, but identified by comparing posterior Bayes odds by Eraker et al.(2003), by excessive skewness in Bates(2000), and by testing a moment condition on options prices in Pan (2002). Hypothesis testing of jumps is a difficult problem, as demonstrated by Khalaf et al. (2003) in the context of the GARCH model; the asymptotic null distribution of the standard test statistics, such as the Wald and likelihood ratio test statistics, is almost intractable, since these test statistics include nuisance parameters, such as jump arrival rate, that is unidentified and cannot be estimated consistently. Davies (1977) first considered the difficulty caused by the presence of nuisance parameters unidentified under the null hypothesis and this problem is sometimes called the Davies problem. A theoretically interesting solution is to construct a new test based on the entire distribution of the original test statistic over a range of values of the unidentified parameter. See Andrews (2001) for the recent development of research in this line.

In this paper we take the conventional approach to this problem with a new technique; we use Dirac's delta function method employed by Kobayashi and Shi (2005) and Shi and Kobayashi (2005) and obtain the Lagrange multiplier test statitics for SV against SVJ and for SVJ against SVCJ free from nuisance parameters unidentified under the null hypothesis; the convolution integral in the derivation of the test statistic can be defined only by regarding the density of jumps with infinitely small variance as Dirac's delta function.

1 TESTING FOR SIMPLE SV AGAINST SV WITH JUMPS IN RETURNS

We first define the simple SV model as

$$y_t = \sigma_t u_t, \quad \theta_t = \alpha + \beta\theta_{t-1} + \sigma v_t, \tag{1}$$

where

$$u_t \sim NID(0,1), \quad v_t \sim NID(0,1), \tag{2}$$
$$\sigma_t^2 = \exp(\theta_t), \quad |\beta| < 1. \tag{3}$$

Then the conditional densities of y_t and θ_t are expressed as

$$g(y_t|\theta_t) \equiv \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{y_t^2}{2\sigma_t^2}\right), \tag{4}$$

$$h(\theta_t|\theta_{t-1}) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\theta_t - \alpha - \beta\theta_{t-1})^2}{2\sigma^2}\right). \tag{5}$$

The density function for the simple SV is obtained by integrating out $\theta_1, \ldots, \theta_T$ from the joint density of $\theta_1, \ldots, \theta_T$ and $y_1, \ldots, y_T$. It is expressed as

$$\begin{aligned} & f(y_1, y_2, \ldots, y_T) \\ &= \int_{-\infty}^{\infty} g(y_T|\theta_T) h(\theta_T|\theta_{T-1}) \\ & \cdots g(y_1|\theta_1) h(\theta_1|\theta_0) d\theta_1 \cdots d\theta_T. \end{aligned} \tag{6}$$

The integration interval $(-\infty, \infty)$ will be suppressed hereafter where there is no fear of ambiguity.

1.1 SV Model with Jumps in Returns

We here assume that a jump in returns occurs with probability p and that the jump size has normal distribution with mean μ and variance λ. The process can be expressed as

$$y_t = \sigma_t u_t + e_t, \quad \sigma_t^2 = \exp(\theta_t), \tag{7}$$
$$\theta_t = \alpha + \beta\theta_{t-1} + \sigma v_t, \tag{8}$$

where the distribution of the jump variable e_t is a mixture of a normal distribution and $\mathbf{0}$ with weights p and $1-p$. It is expressed as

$$e_t \sim \begin{cases} N(\mu, \lambda) & \text{with probability } p, \\ \mathbf{0} & \text{with probability } 1-p, \end{cases} \tag{9}$$

where $\mathbf{0}$ denotes a degenerate distribution with all probability mass at 0. The SVJ model can be expressed as a nonlinear state space model using the measurement and transition equations as follows:

$$\begin{aligned} g_\lambda(y_t|\theta_t) &= \frac{p}{\sqrt{2\pi(\sigma_t^2+\lambda)}} \exp\left(-\frac{(y_t-\mu)^2}{2(\lambda+\sigma_t^2)}\right) \\ &\quad + \frac{1-p}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{y_t^2}{2\sigma_t^2}\right), \\ h(\theta_t|\theta_{t-1}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(\theta_t - \alpha - \beta\theta_{t-1})^2}{2\sigma^2}\right), \\ \sigma_t^2 &= \exp(\theta_t). \end{aligned} \tag{10}$$

The density function of $\tilde{y}_T \equiv (y_1, \ldots, y_T)$ with jumps in returns is expressed as

$$f_\lambda(\tilde{y}_T) = \int g_{\lambda,T} h_T \cdots g_{\lambda,1} h_1 d\theta. \tag{11}$$

where $g_\lambda(y_T|\theta_{T-1}) = g_{\lambda,T}, h(\theta_T|\theta_{T-1}) = h_T$, for example. Evidently, the jump probability p is a nuisance parameter unidentified under the null hypothesis in testing SV against SVJ, because we have the identity

$$f_\lambda(y_1, y_2, \ldots, y_T) = f(y_1, y_2, \ldots, y_T), \tag{12}$$

when $\mu = 0$ and $\lambda = 0$, even if $p > 0$.

1.2 Test Statistic

We here obtain the Lagrange multiplier (LM) test statistic for the null hypothesis $\mu = 0$ and $\lambda = 0$ against the alternative of the SVJ model with unknown parameters $\alpha, \beta, \sigma^2, \lambda$, and μ . We show that the jump probability p, which cannot be estimated under the null hypothesis, is not included in the test statistic, so that p is no longer a nuisance parameter in deriving the distribution of the test statistic.

The Lagrange multiplier test rejects the null hypothesis of the absence of jumps in returns, namely $\mu = 0$ and $\lambda = 0$, when the first derivatives of the logarithm of the likelihood (11) with respect to μ and λ differ sufficiently from zero, because they would be distributed with mean zero under the null hypothesis $\mu = 0$ and $\lambda = 0$. In our notation the score functions are expressed as

$$\partial \log f_\lambda(\tilde{y}_T)/\partial\lambda, \quad \partial \log f_\lambda(\tilde{y}_T)/\partial\mu, \tag{13}$$

evaluated at $\mu = 0, \lambda = 0$, where the first derivatives are evaluated at $\mu = 0$ and $\lambda = 0$ hereafter unless otherwise stated.

First, we have that

$$\begin{aligned}
&\frac{\partial f_\lambda(\tilde{y}_T)}{\partial \lambda} \\
=&\frac{p}{2}\int\left(\frac{y_1^2}{\sigma_1^4}-\frac{1}{\sigma_1^2}\right)g_T h_T\cdots g_1 h_1 d\theta+\cdots \\
&+\frac{p}{2}\int\left(\frac{y_T^2}{\sigma_T^4}-\frac{1}{\sigma_T^2}\right)g_T h_T\cdots g_1 h_1 d\theta,
\end{aligned} \tag{14}$$

noting that

$$\partial g_\lambda(y_t|\theta_t)/\partial\lambda = (p/2)(y_t^2/\sigma_t^4 - 1/\sigma_t^2)g(y_t|\theta_t).$$

Then the score with respect to λ is expressed as

$$\begin{aligned}
&\frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial \lambda} \\
=&\frac{p}{2}\sum_{t=1}^{T}\int\left(y_t^2/\sigma_t^4-1/\sigma_t^2\right) f(\tilde{\theta}_T|\tilde{y}_T)d\theta,
\end{aligned} \tag{15}$$

where the conditional density of $\tilde{y}_T = (y_1,\ldots,y_T), \tilde{\theta} = (\theta_1,\ldots,\theta_T)$ is expressed as

$$f(\tilde{\theta}_T|\tilde{y}_T) = \frac{g_T h_T\cdots g_1 h_1}{f(\tilde{y}_T)}. \tag{16}$$

This conditional density function is obtained by smoothing of the simple SV model. See Hamilton (1996) for a general explanation of smoothing of the nonlinear state space model.

We also have the score function with respect to μ as

$$\frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial \mu} = p\sum_{t=1}^{T}\int\frac{y_t}{\sigma_t^2}f(\tilde{\theta}_T|\tilde{y}_T)d\theta. \tag{17}$$

since we have that

$$\begin{aligned}
&\frac{\partial f_\lambda(\tilde{y}_T)}{\partial \mu} \\
=&p\int\frac{y_T}{\sigma_T^2}g(y_T|\theta_T)h(\theta_T|\theta_{T-1})\cdots g(y_1|\theta_1)h(\theta_1|\theta_0)d\theta+\cdots \\
&+p\int g(y_T|\theta_T)h(\theta_T|\theta_{T-1})\cdots\frac{y_1}{\sigma_1^2}g(y_1|\theta_1)h(\theta_1|\theta_0)d\theta,
\end{aligned} \tag{18}$$

from

$$\partial g_\lambda(y_t|\theta_t)/\partial\mu = g(y_t|\theta_t)y_t/\sigma_t^2. \tag{19}$$

under the null hypohtesis.

Then the LM test statistic for $\mu = 0$ and $\lambda = 0$ is expressed as

$$S_1 = \mathbf{v}_1'\mathbf{I}^{-1}\mathbf{v}_1, \tag{20}$$

where

$$\mathbf{v}_1 = \left(0,0,0,\frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial\lambda},\frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial\mu}\right)' \tag{21}$$

evaluated at $\mu = 0$ and $\lambda = 0$ and $\mathbf{I}$ is the 5×5 Fisher information matrix consisting of

$$\begin{aligned}
I_{ij} &= E\left[-\partial^2\log f_\lambda(y_1,\ldots,y_T)/\partial\zeta_i\partial\zeta_j\right], \\
&(\zeta_1,\ldots,\zeta_5) = (\alpha \quad \beta \quad \sigma^2 \quad \lambda \quad \mu).
\end{aligned}$$

It should be noted here that p in (17) and (15) is cancelled out in the test statistic, because it appears in multiplicative form in the score functions and the Fisher information is variance-covariance matrix of the score functions. The first three elements of $\mathbf{v}_1$ are zero because the ML estimates of α,β,σ^2 are defined by the solution of

$$\frac{\partial \log f(\tilde{y}_T)}{\partial\alpha} = 0, \quad \frac{\partial \log f(\tilde{y}_T)}{\partial\beta} = 0, \quad \frac{\partial \log f(\tilde{y}_T)}{\partial\sigma^2} = 0.$$

Using the BHHH method (Berndt et al. 1974) an elements of the Fisher information matrix is estimated as

$$\sum_{t=1}^{T}\partial\log f(y_t|\tilde{y}_{t-1})/\partial\mu\times\partial\log f(y_t|\tilde{y}_{t-1})/\partial\alpha,$$

for example, as suggested by Hamilton (1996). He also suggested that the first derivative functions of the log likelihood of y_t conditional on $\tilde{y}_{t-1}$ with respect to λ and μ can be evaluated using the formulas

$$\frac{\partial \log f_\lambda(y_t|\tilde{y}_{t-1})}{\partial\lambda} = \frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial\lambda} - \frac{\partial \log f_\lambda(\tilde{y}_{t-1})}{\partial\lambda}, \tag{22}$$

$$\frac{\partial \log f_\lambda(y_t|\tilde{y}_{t-1})}{\partial\mu} = \frac{\partial \log f_\lambda(\tilde{y}_T)}{\partial\mu} - \frac{\partial \log f_\lambda(\tilde{y}_{t-1})}{\partial\mu}, \tag{23}$$

where the unconditional log likelihood of $y_1,\ldots,y_t$ on the right-hand side is evaluated by applying the numerical integration routine used in obtaining (15) and (17). The first derivatives of the conditional log likelihood with respect to α,β and σ^2 can be evaluated easily by differentiating the conditional log likehood function, which is reported in filtering of the nonlinear state-space model.

2 TESTING FOR JUMPS IN VOLATILITY CORRELATED WITH JUMPS IN RETURNS

In this section we derive a Lagrange multiplier test to detect jumps in volatility assuming that jumps in returns occur concurrently with jumps in volatility. The null hypothesis is the SV model with jumps in returns and the alternative hypothesis is the SV model with correlated jumps in returns and volatility (SVCJ) .

2.1 Model

The stochastic volatility model with contemporaneously correlated jumps in returns and volatility have the following state-space representation:

$$y_t = \sigma_t u_t + e_t, \tag{24}$$

$$\theta_t = \alpha + \beta\theta_{t-1} + \sigma v_t + \eta_t, \tag{25}$$

$$\sigma_t^2 = \exp(\theta_t). \tag{26}$$

We assume that jumps in returns and volatility e_t and η_t occur concurrently with probability p and that, conditional on the event that jumps occur, these jumps follows bivariate normal distribution with restriction $\eta_t > 0$. Then the distribution of jumps in volatility follow half-normal distribution, namely the positive part of normal distribution. We will see that this assumption is more convenient than the exponential distribution employed in Eraker et al.(2003) and other papers. For the sake of algebraic convenience we here express the joint density of jumps in returns and volatility conditional on the event that jumps occur as

$$\phi_\lambda(e_t|\eta_t)\phi_\kappa(\eta_t), \tag{27}$$

where

$$\phi_\kappa(\eta_t) = 2(2\pi\kappa)^{-\frac{1}{2}} \exp\left(-\frac{\eta_t^2}{2\kappa}\right), \quad \eta_t > 0, \tag{28}$$

$$\phi_\lambda(e_t|\eta_t) = (2\pi\lambda)^{-\frac{1}{2}} \exp\left(-\frac{(e_t-\mu-\rho\eta_t)^2}{2\lambda}\right). \tag{29}$$

Then the density function of $y_1,\ldots,y_t$ with correlated jumps in returns and volatility is expressed as

$$\begin{aligned} f_\kappa(\tilde{y}_t) =&(1-p)f(\tilde{y}_t) \\ &+p\int g_\kappa(y_t,\theta_t|\theta_{t-1})\cdots g_\kappa(y_1,\theta_1|\theta_0)d\theta, \end{aligned} \tag{30}$$

where

$$\begin{aligned} &g_\kappa(y_t,\theta_t|\theta_{t-1}) = \int g(y_t-e_t|\theta_t)h(\theta_t-\eta_t|\theta_{t-1}) \\ &\phi_\kappa(\eta_t)\phi_\lambda(e_t|\eta_t)de_t d\eta_t. \end{aligned} \tag{31}$$

and the measurement and transition equations, $g(y_t|\theta_t)$ and $h(\theta_t|\theta_{t-1})$, are defined by (4) and (5), respectively.

2.2 Test Statistic

We here obtain the LM test statistic for the null hypothesis of SVJ, namely for $\kappa = 0$ in (28) against the alternative of the SVCJ model with unknown parameters $\alpha,\beta,\sigma^2,\lambda,\kappa,\mu,p$ and ρ. Under the null hypothesis $\kappa = 0$, we have $\eta_t \equiv 0$ identically, so that a test statistic that includes ρ cannot be defined since ρ is unidentifiable and cannot be estimated consistently under the null hypothesis. However, we show that the LM test statistic of our problem is well defined because correlation ρ is cancelled out in the LM test statistic so that the distribution of the test statistic is independent of unidentified ρ.

First, from the likelihood function of SVCJ given in (30) we have

$$\begin{aligned} &\partial f_\kappa(y_1,\ldots,y_T)/\partial\kappa \\ =&p\int \frac{\partial g_\kappa(y_T,\theta_T|\theta_{T-1})}{\partial\kappa} \\ &g_\kappa(y_{T-1},\theta_{T-1}|\theta_{T-2})\cdots g_\kappa(y_1,\theta_1|\theta_0)d\theta+\ldots+ \\ &p\int g_\kappa(y_T,\theta_T|\theta_{T-1})\cdots g_\kappa(y_2,\theta_2|\theta_1)\frac{\partial g_\kappa(y_1,\theta_1|\theta_0)}{\partial\kappa}d\theta. \end{aligned}$$

Noting that the first derivative of the conditional joint density of jumps with respect to κ is

$$\frac{\partial\phi_\kappa(\eta_t)}{\partial\kappa} = (1/2)\phi_\kappa(\eta_t)\left(\frac{\eta_t^2}{\kappa^2}-\frac{1}{\kappa}\right) = (1/2)\frac{\partial^2\phi_\kappa(\eta_t)}{\partial\eta_t^2},$$

we have that

$$\begin{aligned} &\frac{\partial}{\partial\kappa}g_\kappa(y_t,\theta_t|\theta_{t-1}) \\ =&\int g(y_t-e_t|\theta_t)h(\theta_t-\eta_t|\theta_{t-1})\phi_\lambda(e_t|\eta_t)\frac{\partial}{\partial\kappa}\phi_\kappa(\eta_t)de_t d\eta_t \\ =&\frac{1}{2}\int g(y_t-e_t|\theta_t)\frac{\partial^2[h(\theta_t-\eta_t|\theta_{t-1})\phi_\lambda(e_t|\eta_t)]}{\partial\eta_t^2}\phi_\kappa(\eta_t)de_t d\eta_t, \end{aligned}$$

from the formulas of Dirac's delta function (47).

Note that we have

$$\begin{aligned} &\frac{\partial^2}{\partial\eta_t^2}[h(\theta_t-\eta_t|\theta_{t-1})\phi_\lambda(e_t|\eta_t)]\Big|_{\eta_t=0} \\ =&h(\theta_t|\theta_{t-1})\phi_\lambda(e_t|0)\left[\frac{(\theta_t-\alpha-\beta\theta_{t-1})^2}{\sigma^4}-\frac{1}{\sigma^2}\right] \\ &+2h(\theta_t|\theta_{t-1})\phi_\lambda(e_t|0)\left(\frac{\theta_t-\alpha-\beta\theta_{t-1}}{\sigma^2}\right)\left(\frac{e_t-\mu}{\lambda}\right)\rho \\ &+h(\theta_t|\theta_{t-1})\phi_\lambda(e_t|0)\left[\frac{(e_t-\mu)^2}{\lambda^2}-\frac{1}{\lambda}\right]\rho^2. \end{aligned}$$

Then the first derivative of the likelihood of $\tilde{y}_T$ with respect to κ evaluated at $\kappa = 0$ is written as

$$\begin{aligned} &\frac{\partial f_\kappa(\tilde{y}_T)}{\partial\kappa} \\ &=\frac{p}{2}\sum_{t=1}^T\int\left[\frac{(\theta_t-\alpha-\beta\theta_{t-1})^2}{\sigma^4}-\frac{1}{\sigma^2}\right]f_\lambda(\tilde{y}_T)d\theta \\ &+p\rho\sum_{t=1}^T\int\left(\frac{\theta_t-\alpha-\beta\theta_{t-1}}{\sigma^2}\frac{e_t-\mu}{\lambda}\right)f_\lambda(\tilde{e}_T,\tilde{\theta}_T,\tilde{y}_T)d\theta de \\ &+\frac{p}{2}\rho^2\sum_{t=1}^T\int\left[\frac{(e_t-\mu)^2}{\lambda^2}-\frac{1}{\lambda}\right]f_\lambda(\tilde{e}_T,\tilde{y}_T)de, \end{aligned} \tag{32}$$

where the joint density function is defined by

$$\begin{aligned} &f_\lambda(\tilde{\theta}_T,\tilde{y}_T) \equiv \int f_\lambda(\tilde{e}_T,\tilde{\theta}_T,\tilde{y}_T)de \\ &\equiv \int g(y_T-e_T|\theta_T)h_T\phi_\lambda(e_T)\cdots g(y_1-e_1|\theta_1)h_{1,0}\phi_\lambda(e_1)de. \end{aligned}$$

Note that the first term of (32) is zero since it is proportional to the equation that defines the maximum likelihood estimator of σ^2 of SVJ, if we evaluate the term by substituting the maximum likelihood estimator of SVJ, since

$$\frac{\partial h_T}{\partial \sigma^2} = \frac{1}{2}\left[(\theta_t - \alpha - \beta\theta_{t-1})^2/\sigma^4 - 1/\sigma^2\right] h_T.$$

The third term is also zero because this is proportional to the equation that defines the maximum likelihood estimator of λ of SVJ. Then, the hypothesis that $\kappa = 0$ can be tested by checking whether

$$\begin{aligned} &\left.\frac{\partial \log f_\kappa(\tilde{y}_T)}{\partial \kappa}\right|_{\kappa=0} \\ =&p\rho\sum_{t=1}^{T}\int\left(\frac{\theta_t - \alpha - \beta\theta_{t-1}}{\sigma^2}\right)\left(\frac{e_t - \mu}{\lambda}\right) \\ &f_\lambda(\tilde{e}_T|\tilde{\theta}_T, def_\lambda(\tilde{\theta}_T|)d\theta \end{aligned} \tag{33}$$

is sufficiently far from zero; the conditional density of jumps in returns defined by

$$f_\lambda(\tilde{e}_T|\tilde{\theta}_T, \tilde{y}_T) = \frac{f_\lambda(\tilde{e}_T, \tilde{\theta}_T, \tilde{y}_T)}{f_\lambda(\tilde{\theta}_T, \tilde{y}_T)}$$

is written as the product of $f_\lambda(e_t|\theta_t, y_t), t = 1, ..., T$, which is a mixture density of $N(\tilde{\mu}_t, \tilde{\sigma}_t^2)$ and $\delta(e_t)$ with weights $\omega_t \equiv w_{1t}/(w_{1t} + w_{2t})$ and $1 - \omega_t$, as will be seen in Appendix, where

$$\begin{aligned} w_{1t} &= p(2\pi)^{-\frac{1}{2}}(\sigma_t^2 + \lambda)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{(y_t - \mu)^2}{\lambda + \sigma_t^2}\right), \\ w_{2t} &= (1-p)(2\pi\sigma_t^2)^{-\frac{1}{2}}\exp\left(-\frac{y_t^2}{2\sigma_t^2}\right), \\ \tilde{\mu}_t &= \frac{\lambda y_t + \sigma_t^2\mu}{\lambda + \sigma_t^2}, \quad \tilde{\sigma}_t^2 = \frac{\sigma_t^2\lambda}{\lambda + \sigma_t^2}. \end{aligned} \tag{34}$$

Thus, integrating out e_t in (33), we finally have

$$\begin{aligned} &\partial \log f_\kappa(y_1, \dots, y_t)/\partial\kappa \\ =&\frac{p\rho}{\lambda\sigma^2}\sum_{t=1}^{T}\int(\theta_t - \alpha - \beta\theta_{t-1})(\omega_t\tilde{\mu}_t - \mu)\, f(\tilde{\theta}_T|\tilde{y}_T)d\theta. \end{aligned} \tag{35}$$

In this case the Lagrange multiplier test is one-sided, and the test statistic is defined by the

$$\frac{\partial \log f_\kappa(\tilde{y}_t)}{\partial \kappa}$$

divided by the estimated standard deviation, which is the square root of the corresponding element of the inverse of the Fisher information matrix with respect to α, β, σ, μ, λ, p, and κ. The derivation of the Fisher information of the SVCJ model using the BHHH method is similar to that of the SVJ model. In the deriving the test statistic the nuisance parameter ρ is cancelled out by standardization so that the Lagrange multiplier test statistic is independent of the unidentified parameter ρ under the null hypothesis.

3 TESTING FOR JUMPS IN VOLATILITY INDEPENDENT OF JUMPS IN RETURNS

We consider the problem of detecting jumps in volatility in the framework of the SV model with independent jumps in returns and volatility. It is shown that the Lagrange multiplier test statistic cannot be defined in this setting, because the estimated score statistic is zero identically.

3.1 Model

This subsection defines the stochastic volatility model with independent jumps in returns and volatility, which is denoted by SVIJ hereafter. The measurement and transition equations of SVIJ are given as follows:

$$y_t = \sigma_t u_t + e_t, \tag{36}$$

$$\theta_t = \alpha + \beta\theta_{t-1} + \sigma v_t + \eta_t, \tag{37}$$

where e_t and η_t are jumps in returns and volatility, respectively, whose magnitude and arrival time are both independent. Jumps in returns occur with probability p in the same manner as given in (2.3) and jumps in volatility occurs with probability q and the size of jumps in volatility η_t follows a half-normal distribution. This model is different from the setting in the previous section in that the jumps in returns and volatility occurs independently and their jump sizes are correlated. However, we have the same result that the test to detect jumps in volatility cannot be defined even if there is no jumps in returns; we have only to replace $g_\lambda(\ )$ with $g(\ )$ in the following algebra.

The density function of θ_t conditional on θ_{t-1} is expressed as

$$\begin{aligned} h_\tau(\theta_t|\theta_{t-1}) &= (1-q)h(\theta_t|\theta_{t-1}) \\ +q&\int_0^\infty h(\theta_t - \eta_t|\theta_{t-1})\phi_\tau(\eta_t)d\eta_t, \end{aligned} \tag{38}$$

where the conditional density of η_t when a jump occurs is

$$\phi_\tau(\eta_t) = \frac{2}{\sqrt{2\pi\tau}}\exp\left(-\frac{\eta_t^2}{2\tau}\right). \tag{39}$$

The likelihood function the SV process with independently arriving jumps in returns and volatility is expressed as

$$\begin{aligned} f_\tau(\tilde{y}_T) = \int & g_\lambda(y_T|\theta_T)h_\tau(\theta_T|\theta_{T-1}) \\ &\cdots g_\lambda(y_1|\theta_1)h_\tau(\theta_1|\theta_0)d\theta, \end{aligned} \tag{40}$$

where $g_\lambda(y_t|\theta_t)$ is defined in (10).

3.2 Test Statistic

We here consider the LM test statistic for the null hypothesis of SVJ, namely for $\tau = 0$ in (39), against the alternative of the SVIJ model with unknown parameters $\alpha, \beta, \sigma^2, \lambda, \mu$, and τ. We show that the derivative of the logarithm of the likelihood of SVIJ (40) with respect to τ is identically zero so that the LM test for the null of SVJ against SVIJ cannot be defined. In this subsection we evaluate expressions at $\tau = 0$ unless otherwise stated, while the other parameters $\alpha, \beta, \sigma^2, \lambda,$ and μ are estimated by ML.

From (40), we have

$$\begin{aligned}&\frac{\partial f_\tau(\tilde{y}_T)}{\partial \tau}\\ =&\int g_\lambda(y_T|\theta_T)\frac{\partial h_\tau(\theta_T|\theta_{T-1})}{\partial \tau}\cdots g_\lambda(y_1|\theta_1)h_\tau(\theta_1|\theta_0)d\theta\\ &+\cdots+\int g_\lambda(y_T|\theta_T)h_\tau(\theta_T|\theta_{T-1})\cdots g_\lambda(y_1|\theta_1)\frac{\partial h_\tau(\theta_1|\theta_0)}{\partial \tau}d\theta.\end{aligned} \tag{41}$$

We have that

$$\begin{aligned}&\frac{\partial h_\tau(\theta_t|\theta_{t-1})}{\partial \tau}\\ =&\frac{q}{2}h(\theta_t|\theta_{t-1})\left[\frac{(\theta_t-\alpha-\beta\theta_{t-1})^2}{\sigma^4}-\frac{1}{\sigma^2}\right]\end{aligned} \tag{42}$$

at $\tau = 0$, from the equality

$$\frac{\partial \phi_\tau(\eta_t)}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \phi_\tau(\eta_t)}{\partial \eta_t^2} \tag{43}$$

and (47) of the formulas of Dirac's delta function.

Then, at $\tau = 0$, we have that

$$\begin{aligned}&\partial \log f_\tau(\tilde{y}_T)/\partial\tau|_{\tau=0}\\ =&\frac{q}{2}\sum_{t=1}^{T}\int\left[\frac{(\theta_t-\alpha-\beta\theta_{t-1})^2}{\sigma^4}-\frac{1}{\sigma^2}\right]f_\lambda(\tilde{\theta}_T|\tilde{y}_T)d\theta.\end{aligned} \tag{44}$$

This estimated score is identically zero and hence cannot be used as a test statistic, since it is proportional to the equation that defines the ML estimator of σ^2 of SVJ.

4 APPENDIX

4.1 Dirac's Delta function

Dirac's delta function $\delta(\cdot)$ is a degenerate density function with all probability mass at 0. It can be easily shown by integration by parts that

$$\int k(x)\delta(x)dx = k(0), \tag{45}$$

$$\int k(x)\frac{d}{dx}\delta(x)dx = -\frac{d}{dx}k(0), \tag{46}$$

$$\int k(x)\frac{d^2}{dx^2}\delta(x)dx = \frac{d^2}{dx^2}k(0) \tag{47}$$

for an arbitrary regular function $k(x)$. Detailed discussion about Dirac's delta function can be found in most textbooks of the Fourier transformation. See Bracewell (1999), for example. The normal density function with infinitely small variance, such as $\phi_\kappa(\eta_t)$ when κ is infinitely small, is a typical example of Dirac's delta function. For the usage of Dirac's delta function in different contexts, see Peers (1971), Kobayashi (1991), and Kobayashi and Shi (2005).

4.2 Conditional Distribution of Variables of SV with Jumps in Returns

The purpose of this subsection is to derive the conditional density of e_t given $y_1, \ldots, y_T$ and θ_t in (33). In our problem, y_t and θ_t have all information about e_t and hence

$$f(e_t|\tilde{y}_t, \theta_t) = f(e_t|y_t, \theta_t).$$

We also have that

$$\begin{aligned}&f(e_t, y_t|\theta_t) = f(e_t)f(y_t|e_t, \theta_t)\\ =&[p\phi(e_t;\mu,\lambda)+(1-p)\delta(e_t)]\phi(y_t-e_t;0,\sigma_t^2)\\ =&p\phi(e_t;\mu,\lambda)\phi(y_t-e_t;0,\sigma_t^2)+(1-p)\phi(y_t;0,\sigma_t^2)\delta(e_t).\end{aligned} \tag{48}$$

where $\phi(x;\mu,\sigma^2)$ denotes normal density function with mean μ and variance σ^2. After some algebra, we have that

$$\begin{aligned}&f(e_t|\theta_t, y_t) = \frac{f(e_t, y_t|\theta_t)}{f(y_t|\theta_t)}\\ =&\frac{w_{1t}(2\pi\tilde{\sigma}_t^2)^{-\frac{1}{2}}\exp\left(-(e_t-\tilde{\mu}_t)^2/(2\tilde{\sigma}_t^2)\right)+w_{2t}\delta(e_t)}{w_{1t}+w_{2t}},\end{aligned} \tag{49}$$

where

$$\tilde{\mu}_t = \frac{\lambda y_t+\sigma_t^2\mu}{\lambda+\sigma_t^2}, \quad \tilde{\sigma}_t^2 = \frac{\sigma_t^2\lambda}{\lambda+\sigma_t^2},$$

$$w_{1t} = p(2\pi)^{-\frac{1}{2}}(\sigma_t^2+\lambda)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{(y_t-\mu)^2}{\lambda+\sigma_t^2}\right),$$

$$w_{2t} = (1-p)(2\pi\sigma_t^2)^{-\frac{1}{2}}\exp\left(-\frac{y_t^2}{2\sigma_t^2}\right).$$

5 EMPIRICAL EXAMPLES AND MONTE CARLO EXPERIMENT

We first calculate our test statistics for the daily return of the S&P index from 1 January 1985 to 15

December 1988 (T=1000). The sample size is 1000. The estimated β is 0.958, and the LM test statistic for SV against SVJ is 8.91, which is bigger than upper five percentile of $\chi^2(1)$ so that the null hypothesis of no jumps in returns is rejected at significance level 0.05. On the other hand, the value of the LM test statistic for the same series from 15 December 1988 to 1 December 1992 (T=1000) is 2.34 so that the null hypothesis of no jumps in returns cannot be rejected. This contrasting result is natural because the former sample includes Black Monday, October 19, 1987.

We have performed a small Monte Carlo experiment with sample size 1000 and number of iterations 400 only for the LM test for SV against SVJ because one iteration of the LM test for SVJ against SVCJ takes several hours even using GAUSS.

Figure 1 shows the histogram of the empirical distribution of the LM test statistics for SV against SVJ when $\alpha = 0, \beta = 0.9, \sigma = 0.4$ in the data generation process (1).

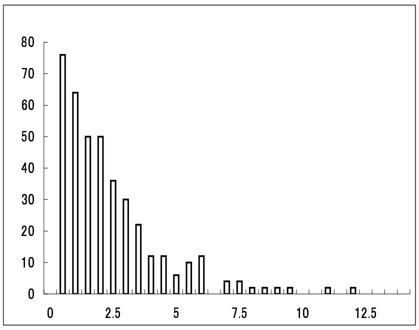

Figure 1. Empirical Distribution of the LM Test Statistic for SV against SV with Jumps in Returns, $\alpha = 0, \beta = 0.9, \sigma = 0.4$, T=1000, Number of Iterations=400

As far as Figure 2 shows, the deviation the actual distribution from the $\chi^2(2)$ distribution is not substantial.

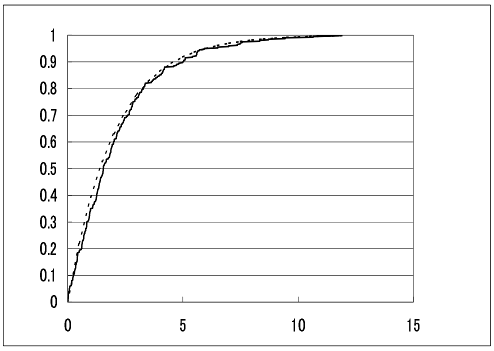

Figure 2. Cumulative Distribution of $\chi^2(2)$ (Dotted Line) and Empirical Distribution of the LM Test Statistic for SV against SVJ (Solid Line), $\alpha = 0, \beta = 0.9, \sigma = 0.4$, T=1000, Number of Iterations=400

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