Using Analytical Solutions For Homogenous Soils To Assess Numerical Solutions For Layered Soils

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EXTENDED ABSTRACT

Analytical solutions for non-steady flow are an important aspect of mathematical modeling in all fields of computational science. An analytical solution provides an exact solution for a specific (simplified) test case, which then can be used to test and verify numerical solutions. Within soil physics, there has been a multitude of analytical solutions that model transient flow through one-dimensional homogenous soil profiles under various flow conditions. For homogenous soils, analytical solutions exist for realistic soil types (i.e. non-linear hydraulic functions) and for coupled solute and water transport. However, for layered soils, there has only been one analytical solution for non-steady flow. Even though this solution has been useful for testing numerical schemes, the disadvantages of the solution are 1) it is lengthy, complex and difficult to program; 2) is only valid for a particular form of the hydraulic functions with a constant hydraulic diffusivity (D); and 3) one of the key soil parameters is constant across soil layers.

To overcome these limitations, we will use a transformation technique to transform an analytical solution for water flow in a homogenous soil to obtain an analytical solution for an idealized layered soil profile. The idealized analytical solution arises from only transforming part of the solution over a selected segment of the spatial domain. For this study, we examine 1) a linear transformation of the solution variable (θ) and the spatial coordinate system (z) and 2) a non-linear transformation of the solution variable (θ). As a starting point, we will use a simple analytical solution developed by Clothier et al. (1981), which models constant flux infiltration into a field soil: Bungendore fine sand. This soil is a special case since experimentally, it was shown that D is near constant and K is approximately quadratic. Under these properties (1) reduces to the well-known Burgers' equation. Note that under the non-linear transformation the new hydraulic functions, that is, D and the hydraulic conductivity (K) will take the power law form, which is commonly used within soil physics.

Both these approaches will be used to test the accuracy of a numerical solution from the Method of Lines (MoL) for infiltration into layered soils for the water content (θ) Richards' Equation. In addition, we will also use the notion of a relative gradient to assess the affect a given transformation has on the gradients within the system. Ideally, a parameter within transformation functions could be used to increase (or decrease) the gradients within the system to further test numerical schemes under varying flow scenarios.

It will be shown that, for both transformations, the MoL solution showed excellent agreement with the analytical solution for the idealized layered systems. For the linear case, the linear transformation of θ had no effect on the flow dynamics of the original homogenous solution. A linear transformation of z had the effect of stretching or shrinking the underlying soil layer, which will artificial change the steepness of gradients within the system as long as Δz remains constant. However, this affect could also be achieved without the use of transformations by varying Δz within the system. For the non-linear case, the transformation did have an affect on the flow behaviour in layer 2. However, the overall affect of the transformation was to ease the gradients within the system. Ideally, it would be beneficial to derive a non-linear transformation that will can decrease and increase gradients within the system to fully test numerical schemes for water flow through layered soils.

For future work, other non-linear transformations will be explored, in conjunction with other analytical solutions, to obtain a more appropriate transformation to test numerical solution for layered soils under various flow conditions.

1. INTRODUCTION

The accurate simulation of water flow through soils is an important environmental problem and has applications in various fields such as agriculture, waste management and water management. For some cases, soil profiles can be considered homogenous but in most cases, soil profiles are heterogeneous and can consist of distinct soil layers (Raats, 2001; Matthews et al. 2004). Vertical water flow through a one-dimensional unsaturated homogenous soil profile is described by Richards' Equation (Clothier et al., 1981) as

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right)-\frac{dK}{d\theta}\frac{\partial\theta}{\partial z}, \qquad (1)$$

where θ is water content [L^3L^{-3}], D is hydraulic diffusivity [LT^{-2}], K is hydraulic conductivity [LT^{-1}], z is vertical coordinate positive downwards with the soil surface at $z = 0$ [L] and t is time [T]. Note that water flux (q) is defined as

$$q=-D\frac{\partial\theta}{\partial z}+K\ . \qquad (2)$$

For vertical flow through a layered soil system, (1) can be used to describe the flow dynamics in each soil layer with each layer having soil specific hydraulic functions. For a multilayered soil, this results in a system of partial differential equations (PDEs) with each PDE being coupled to its surrounding layers via the interface boundary condition. The condition at the interface is the continuity of flux (q) and hydraulic pressure (h) so that $q_n = q_{n+1}$ and $h_n = h_{n+1}$, where n is the soil counter. Generally, (1) is not used to model flow through layered soils since θ is discontinuous across soil layers. Recently, Matthews et al. (2004) showed that (1) is applicable to layered soils by developing an iterative solution to handle the discontinuity at the interface, which is an approach that is used here.

For most soil hydraulic functions (i.e. D and K), (1) is highly non-linear, hence, analytical solutions can only be obtained for simplified test cases. As a result, numerical solutions are generally sought for more complex flow scenarios. Nevertheless, analytical solutions are an important aspect of mathematical modeling since 1) exact solution can highlight important flow behaviour and 2) analytical solutions are used to test the accuracy of numerical solutions, which is the focus of this paper. Within the soil physics literature, there are several analytical solutions designed for specific test problems. These include constant flux infiltration (Clothier et al., 1981; Sander et al., 1988), two-phase flow (Raats, 2001) and, more recently, coupled solute and water transport (Sander et al., 2005). One of the simplifying assumptions for most analytical solutions is that the soil profile must be homogenous. In fact, Srivastava and Yeh (1991) have published the only analytical solution for transient flow through a layered soil profile. Even though this solution has been useful for testing various numerical schemes (Matthews et al., 2004), there are three main disadvantages associated with this analytical solution 1) it is lengthy, complex and difficult to program; 2) is only valid for a particular form of the hydraulic functions with a constant hydraulic diffusivity (D); and 3) one of the key soil parameters is constant across soil layers.

Given the paucity of analytical solutions for layered soils, this paper will develop a technique that uses existing analytical solutions to test numerical solutions for water flow through layered profiles. As a starting point, we will use a simple analytical solution developed by Clothier et al. (1981), which models constant flux infiltration into a field soil: Bungendore fine sand. This soil is a special case since experimentally, it was shown that D is near constant and K is approximately quadratic. Under these properties (1) reduces to the well-known Burgers' equation.

2. BURGERS' EQUATION

The reduced form of Richards' Equation is given by Clothier et al. (1981) as

$$\frac{\partial\theta}{\partial t}=D\frac{\partial^2\theta}{\partial z^2}-A(\theta-\theta_r)\frac{\partial\theta}{\partial z}, \qquad (3)$$

where D and A are constants, θ_r is the residual water content and $dK/d\theta = A(\theta-\theta_r)$. Note that $K(\theta)=A(\theta-\theta_r)^2/2$ giving $K(\theta_r) = 0$. Clothier et al. (1981) derived an analytical solution of (3) for an infinite soil profile under a constant surface flux boundary condition and a constant θ initial condition. The initial and boundary condition are given by

$$\theta_i=\theta_r,\quad z>0,\quad t=0\ , \qquad (4)$$

$$q_0=K(\theta_0),\quad z=0,\quad t\geq 0\ . \qquad (5)$$

Details of the analytical solution can be found at Clothier et al. (1981). Note that Clothier et al. (1981) showed, for the Burgendore sand, that the analytical solution agreed well with experimental data under the above conditions.

3. TRANSFORMATION TO LAYERED SOILS

We are interested in transforming the homogenous analytical solution so that it is applicable to the layered soil system shown in Figure 1.

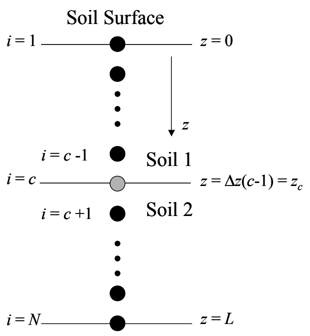

Figure 1. Grid system for 1D layered soil profile.

For this study, we will accept the analytical solution as the exact solution within the soil profile until z = z_c. For $z_c \le z \le L$, the solution will be transformed to obtain a different flow behaviour in the underlying soil. This study will consider two transformations 1) linear transformation of both θ and z, and 2) non-linear transform of θ to highlight disadvantages and advantages of each approach. Note that these transformations will create an artificial soil layer with hydraulic properties determined by the transformation functions, which will result in a discontinuity of θ at the interface.

3.1. Linear Transformations of θ and z

For the linear transformation, we will scale θ and z by the constants μ and λ, respectively so that

$$\hat{\theta} = \mu\theta \, , \tag{6}$$

$$\hat{z} = \lambda z \, . \tag{7}$$

Applying (6) and (7) to (3) yields

$$\frac{\partial \hat{\theta}}{\partial t} = D\lambda^2 \frac{\partial^2 \hat{\theta}}{\partial \hat{z}^2} - A\lambda \left(\frac{\hat{\theta}}{\mu} + \theta_r \right) \frac{\partial \hat{\theta}}{\partial \hat{z}} \, . \tag{8}$$

Therefore, the governing equations for modeling flow through the idealized layered soil system (Figure 1) are given by (3) for $0 \le z \le z_c$ and (8) for $z_c \le z \le L$. The surface boundary condition is still be given by (5) and the initial condition is now given by

$$\begin{aligned} &\theta_i = \theta_r, \quad 0 < z \le z_c, \quad t = 0 \\ &\theta_i = \hat{\theta}_r = \mu\theta_r, \quad z_c \le z \le L, \quad t = 0. \end{aligned} \tag{9}$$

The boundary conditions at the interface, $h(\theta) = \hat{h}(\hat{\theta})$ and $q = \hat{q}$, can be written in terms of θ as (6) and

$$\frac{\partial \theta}{\partial z} = \frac{\lambda}{\mu} \frac{\partial \hat{\theta}}{\partial \hat{z}} \, , \tag{10}$$

respectively, where

$$\hat{q} = D \frac{\lambda}{\mu} \frac{\partial \hat{\theta}}{\partial \hat{z}} + K \, . \tag{11}$$

3.2. Non-linear Transformation of θ

For the non-linear transformation, θ is scaled by some non-linear function f and is given by

$$\bar{\theta} = f(\theta) \, . \tag{12}$$

Applying (12) to (3) yields

$$\frac{\partial \bar{\theta}}{\partial t} = \frac{d\bar{\theta}}{d\theta} \frac{\partial}{\partial z} \left(D \frac{d\theta}{d\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z} \right) - \left(A\theta(\bar{\theta}) + B \right) \frac{\partial \bar{\theta}}{\partial z} \, , \tag{13}$$

where $\bar{\theta}$ is the non-linear transformed variable. Again, the governing equations for the layered system are given by (3) and (13) over $0 \le z \le z_c$ and $z_c \le z \le L$, respectively. The surface boundary condition given by (5) and the initial condition at $t = 0$ for $z_c \le z \le L$ is given by

$$\theta_i = \bar{\theta}_i = f(\theta) \, , \tag{14}$$

with the initial condition for $0 < z \le z_c$ still given by (9). Following the same procedure in section 3.1, the boundary conditions at the interface at $z = z_c$ for $t > 0$ is given by (12) and

$$\frac{\partial \theta}{\partial z} = \frac{d\theta}{d\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z} \, . \tag{15}$$

For this study, we will assume that the relationship between θ and $\bar{\theta}$ is given by

$$\theta = \frac{\bar{\theta}_s - \bar{\theta}_r}{\eta + 1} \left(\frac{\bar{\theta} - \bar{\theta}_r}{\bar{\theta}_s - \bar{\theta}_r} \right)^{\eta+1} + \theta_r \, , \tag{16}$$

which gives

$$\frac{d\theta}{d\bar{\theta}} = g(\bar{\theta}) = \left(\frac{\bar{\theta} - \bar{\theta}_r}{\bar{\theta}_s - \bar{\theta}_r} \right)^{\eta} \, , \tag{17}$$

where η is a constant, $\bar{\theta}_r$ is the scaled residue water content and $\bar{\theta}_s = \bar{\theta}_r + (\theta_s - \theta_r)(\eta + 1)$ is the scaled saturated water content. Note that unrealistic values for water content can occur from this transformation i.e. $\bar{\theta} > 1$. However, η (which must be > 0) can be bounded on the positive domain to achieve a more realistic range for values for $\bar{\theta}$. Since this paper is concerned with testing numerical schemes for layered media, we have allowed η to remain unbounded.

Applying (16) and (17) to (13), results in the hydraulic functions in (13) resembling a power law D and K. This form of the hydraulic functions has been used extensively within soil physics (Raats, 2001). It should also be noted that both the linear and non-linear transformations results in a discontinuity at the interface.

4. METHOD OF LINES

The numerical technique used here is the Method of Lines (MoL). In particular, we used the MoL template developed by Lee et al. (2004) in Matlab to solve the transformed layered systems. Fundamentally, the MoL discretises the spatial part of a PDE while keeping the time part continuous. This results in a system of ordinary differential equations (ODEs) with each ODE representing the time evolution of the flow dynamics at a particular node within the system. Within the template, we use the Matlab function ODE15s to integrate the system of ODEs.

The discretisation technique used in the MoL template is a fourth-order finite differencing (FD) scheme based on an approach by Schiesser (1991). Unlike conventional scheme, Schiesser's approach uses a series of backward and forward FD equations to account for the boundaries within the domain instead of fictitious nodes. Boundary conditions are incorporated into the scheme by imposing the condition on an appropriate vector within the model depending on the type of boundary condition, that is, Neumann or Dirchlet. This feature, coupled with the grid construction in Figure 1, is of particular importance for the interface boundary condition since it allows the flow dynamics of each soil to be estimated at the interface (Matthews et al., 2004b). This approach was shown by Matthews et al. (2004a; 2004b) to handle the discontinuity in θ at the interface.

4.1. Interface Boundary condition

Within the MoL, the interface boundary condition for both cases will be handled in a similar fashion to Matthews et al. (2004b). Essentially, a direct solution for θ is sought by recasting the q continuity equation (e.g. (10)) solely in terms of θ. In Matthews et al. (2004b), this is achieved by using the continuity of h condition to derive a relationship between the water contents of the two connecting soils. For this study, this relationship is already given by (6) and (12). After applying the FD scheme to the derivative terms, this procedure resulted in a non-linear equation for θ, which was solved using Newton's Method (Matthews et al., 2004b).

For the linear transformed case, substituting the FD scheme at $z = z_c$ into (10) and using the condition specified by (6), reduces the interface boundary condition to

$$\Delta_{\mathrm{b}}(\theta) - \lambda \Delta_{\mathrm{f}}(\theta) = 0\,, \qquad (18)$$

where Δ_{f} and Δ_{b} are forward and backward FD schemes respectively. This results in a linear equation which is readily solved for θ and, consequently $\hat{\theta}$, at each time step. For the non-linear transformation, applying the FD scheme and the condition given by (12), (15) can be written solely in terms of θ as follows

$$\Delta_{\mathrm{b}}(\theta) - g(f(\theta))\Delta_{\mathrm{f}}(f(\theta)) = 0\,. \qquad (19)$$

Equation (19) is non-linear and is solved for θ in the MoL using Newton's Method. Note that the iterative scheme is terminated when $\left|\theta^{j+1} - \theta^{j}\right| \le 10^{-10}$, where j is the iteration counter.

5. RELATIVE GRADIENT

Matthews et al. (2005) explored the effect transformations had on (1) via the notion of a relative gradient. A relative gradient is defined as the magnitude of a gradient, relative to the magnitude of the solution variable it is acting on and is given by

$$RG_{\theta} = \frac{\partial\theta/\partial\xi}{\theta}\,, \qquad (20)$$

where ξ is any coordinate system. When transforming Richards' Equation in terms of $\bar{\theta}$ (or $\hat{\theta}$), if $RG_{\theta} > RG_{\bar{\theta}}$ the transformation will have a diminishing effect on spatial and/or temporal gradients within the system. Conversely, if $RG_{\theta} < RG_{\bar{\theta}}$ will increase the effect of gradients. From these conditions, we can define a relative gradient ratio (ζ) as

$$\zeta = \frac{RG_{\theta}}{RG_{\bar{\theta}}} = \frac{d\theta}{d\bar{\theta}}\frac{\bar{\theta}}{\theta}\,. \qquad (21)$$

so that $\zeta > 1$ gradients decrease and for $\zeta < 1$ gradients increase in the transformed space.

6. RESULTS

Water flow through the linear and non-linear transformed systems will be simulated for a 30 cm soil profile with the interface situated at 5cm. The spatial step size (Δz) will be kept consistent at 0.25 cm for all simulations. Each model will be run for a period of t = 12 mins, which gives enough time for water to infiltrate into the next layer. The soil parameters for the fine sand where set at $\theta_s = 0.385$ cm^3/cm^3, $\theta_r = 0.05$ cm^3/cm^3, $D = 1$ cm^2/min and $A = 5$ cm/min except where otherwise specified. Also, $\theta_0 = 0.25$ which provides a constant flux given by (5). The soil properties of the underlying idealized soil are determined by the transformation functions outlined in section 3. The non-linear transformed system also required the specification of $\overline{\theta}_r$, which was set at 0.01 cm^3/cm^3.

Solutions of the two idealized layered soil systems are compared against the analytical solution from Clothier et al. (1981). To compare the numerical and analytical solutions, relative error (RE) will be calculated for each node as

$$\mathrm{RE}_i = \frac{\left|\theta_i^{\mathrm{num}} - \theta_i^{\mathrm{exact}}\right|}{\theta_i^{\mathrm{exact}}}, \qquad (22)$$

where θ_i^{num} and $\theta_i^{\mathrm{exact}}$ are the water content from the numerical and analytical solution, respectively and i is the node counter. Note that for $z \geq z_c$, $\theta_i^{\mathrm{exact}}$ is transformed by the appropriate transformation functions before RE is calculated. As an additional measure, average RE is calculated over both layers separately so that

$$\mathrm{ARE} = \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} \mathrm{RE}\, dz, \qquad (23)$$

where z_1 and z_2 are the upper and lower bounds of each layer i.e. for layer 1, $z_1 = 0$ (surface) and $z_2 = z_c$. Note that the integral is calculated in discrete form using Simpson's rule.

6.1 Linear transformation

The linearly transformed idealized layered system was run for several values of λ = 0.2, 0.5, 1 and 2 and for μ = 0.5 and 2. For $\lambda = 1$, the geometry specified above will not be altered. For $\lambda < 1$, the bottom layer will shrink by a factor of λ while for $\lambda > 1$ the bottom layer will be stretched. This results in $\hat{z}_L$ being determined by $\hat{z}_L = \lambda(z_L - z_c)$. Also, the parameter μ controls the extent of the discontinuity at the interface through (6).

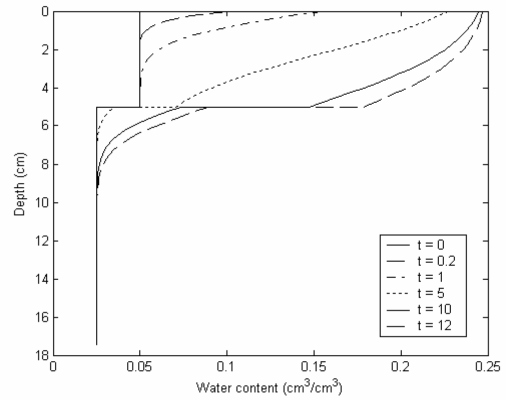

Figure 2. Water content (θ) profiles over depth for the non-linear transformation given by (16) with $\lambda = \mu = 0.5$.

Figure 2 shows a plot of water content over depth for various time intervals until the final time of 12 minutes for $\lambda = \mu = 0.5$. At earlier times, the effective of a stronger convective flow is evident, which is smeared over time by dispersion. This makes sense since $q_0 = D = 1$ so that the magnitude of the conductivity K is less than the magnitude of D for $\theta < \theta_0$. In addition, Figure 2 shows the effect of $\mu = 0.5$ with a distinct discontinuity at the interface at z = 5cm, and the extent of which is determined by (6) with the profile shifting to the left within a smaller range of θ. Also, $\lambda < 1$, Figure 2 shows that the second layer has shrunken by a factor of 0.5 resulting in the total length of the profile being 17.5 cm.

Table 1. Relative error at the interface ($z = z_c$) and average relative error over top (layers 1) and bottom (layer 2) for various λ and μ values.

λ	μ	RE $z = z_c$	ARE layer 1	ARE layer 2
0.2	0.5	9.86x10^{-4}	4.30x10^{-4}	2.35x10^{-3}
0.5	0.5	1.29x10^{-4}	1.96x10^{-5}	7.10x10^{-5}
1	0.5	7.77x10^{-6}	4.60x10^{-6}	1.99x10^{-5}
2	0.5	1.41x10^{-5}	5.64x10^{-6}	1.94x10^{-5}
0.2	2	9.86x10^{-4}	4.30x10^{-4}	2.35x10^{-3}

Table 1 shows RE at the interface and ARE over the top layer (layer 1) and the bottom layer (layer 2) for each value of λ and μ. It is evident that as $\lambda < 1$, RE and ARE in both layers increase for all values of μ considered. Given that Δz is kept constant across all simulations, the increase in error is caused by shrinking the flow dynamics over a smaller range of z. Essentially, as $\lambda < 1$, the grid system in the second layer becomes coarser since there are less nodes describing the same flow behaviour. For example, for $\lambda = 1$ there are 101 nodes in the second layer while for $\lambda = 0.2$ there are only 51 nodes. The coarser grid system will artificial create steeper gradients and increase the truncation error in the finite differencing schemes causing the increase in error. The increase in ARE in the first layer is directly caused by the interface boundary condition since the finite differencing schemes are link at the interface through (18). Note that RE at $z = z_c$ is consistently higher than the ARE over the first layer. For $\lambda > 1$, the ARE in the second layer continues to decrease, which is to be expected since the stretching of the second layer will resulting in a more nodes describing the same flow dynamics i.e. a finer grid. However, the RE at the interface and consequently ARE in the first layer has increased, which seems to be counterintuitive. Interestingly, μ has no effect on ARE over the soil layers and RE at the interface, which is highlighted in Table 1 for $\lambda = 0.5$ and $\mu = 0.5$ and 2.

6.2 Non-linear transformation

The non-linear transformed idealized layered soil system, was run for various values of $\eta = 0, 0.5, 2, 5$ for the test case given above. For $\eta = 0$, (16) will reduce to a linear condition at the interface given by $\theta = \bar{\theta} - (\bar{\theta}_r - \theta_r)$ resulting in a wetting profile that is very close to the original homogenous profile. For $\eta < 0$, the water content will be stretched out in to a larger range of values shifting the discontinuity to the right. For $\eta > 0$, the water content range will shrink and shift the discontinuity to the left towards zero. In addition, simulations for A = 10 cm/min were run for all values of η to examine the effect of increasing the convective flow.

Table 2. Relative error at the interface ($z = z_c$) and average relative error over layers 1 and 2 for various η values.

A	η	RE $z = z_c$	ARE layer 1	ARE layer 2
5	0	7.77x10^{-6}	4.60x10^{-6}	2.01x10^{-5}
	0.5	6.96x10^{-6}	4.45x10^{-6}	1.43x10^{-5}
	2	6.61x10^{-6}	4.55x10^{-6}	7.28x10^{-6}
	5	6.40x10^{-6}	4.68x10^{-6}	3.66x10^{-6}
10	0	2.00x10^{-5}	3.64x10^{-6}	1.27x10^{-3}
	0.5	1.79x10^{-5}	3.84x10^{-6}	9.12x10^{-4}
	2	1.54x10^{-5}	4.07x10^{-6}	4.71x10^{-4}
	5	1.40x10^{-5}	4.22x10^{-6}	2.37x10^{-4}

Table 2 shows RE and ARE over layer 1 and 2 for the non-linear transformed case for all values of η and for $A = 5$ and 10 cm/min. In Table 2, it is evident the MoL model compares well with the analytical solution with RE and ARE being within the order of 10^{-5}. However, this is not the case for $A = 10$ cm/min where ARE over the layer 2 is within the order of 10^{-3}. This increase in error is to be expected since sharper wetting fronts will occur. For $A = 10$ cm/min, the RE at $z = z_c$ has also increased as compare to $A = 5$ cm/min but not to the extent as the ARE for the second layer. Note that ARE in layer 1 has decreased from $A = 5$ to 10 cm/min but this is caused by the top layer reaching steady state for A = 10 cm/min.

For $A = 5$ and 10 cm/min, Table 2 also highlights that as η increases RE at $z = z_c$ and ARE over layer 2 decreases. Interestingly, ARE over layer 2 almost changes by an order of magnitude while RE at $z = z_c$ undergoes a small change from $\eta = 0$ to 5. For layer 1, ARE also undergoes a small change except that ARE increases as η increases. Note that, for $A = 5$ cm/min, there is an initial decrease in ARE from $\eta = 0$ to $\eta = 0.5$ but then increases steadily until $\eta = 5$. Given the non-linear nature of the transformation, the above behavior is not immediately obvious and will be discussed in the next section in terms of relative gradients.

7. DISCUSSION

To gain an understanding of the error behaviour exhibited in Table 2, we will use (21) to examine how the non-linear transformation affects the relative gradients within the system. Substituting (16) and (17) into (21) provides a function for ζ in terms of either θ or $\bar{\theta}$.

Figure 3 shows a plot of ζ over the full wetting range in terms of θ i.e. from θ_r to θ_s. From Figure 3, it is evident that as η increase the magnitude of ζ also increases for the majority of the wetting range. Note that of $\zeta > 1$ the transformation will decrease the effects of relative gradients within the system. Therefore, as η increase, the wetting gradients within the second layer will decrease resulting in the decrease in ARE. This is also the case for RE at $z = z_c$. The increase in ARE for the first layer is counterintuitive since the first layer is only affected by the transformation through the interface boundary condition. However, the ARE stays relatively stable for all values of η between $A = 5$ and 10, which demonstrates that the affect is minimal for the cases considered. Note that applying the linear transformation to the relative gradient condition gives $\zeta = \lambda$. This confirms that μ has no affect on the solution and λ decrease the effects of gradients has λ increases.

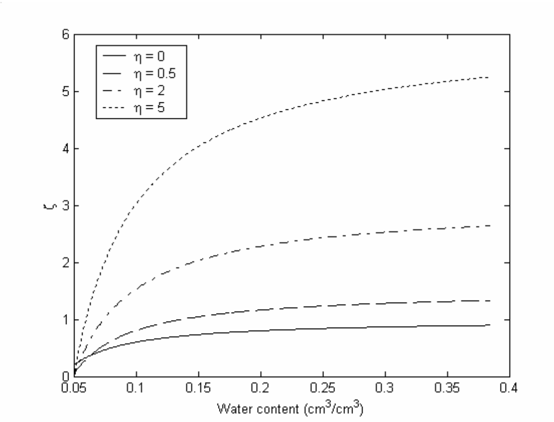

Figure 3. Relative gradient ratio (ζ) vs water content (θ) over the full wetting range for $\eta = 0, 0.5, 2$ and 5.

8. CONCLUSIONS

This paper outlines a method that utilises analytical solution for water flow through homogenous soil profiles to test numerical solution for layered soils. This is achieved by transforming a segment of the soil profile to give an artificial soil layer. The artificial layer was created by 1) performing a linear transformation on θ and z and 2) a non-linear transformation on θ. It was shown that a linear transformation of θ had no effect on the flow dynamics of the homogenous solution. A linear transformation of z had the effect of stretch or shrinking the underlying soil layer, which will artificial change the steepness of gradients within the system as long as Δz remains constant. However, this affect could also be achieved without the use of transformations by varying Δz within layer 2. We showed that the non-linear transformation did have an affect on the flow behaviour in layer 2. However, the overall effect of (16) was to ease the effect of gradients within the system. For both transformations, the MoL solution showed excellent agreement with the analytical solution but lost accuracy as the convective flow was increased. For future work, other non-linear transformation will be explored, in conjunction with other analytical solutions, to obtain an appropriate transformation to test numerical solution for layered soils under various flow conditions.

9. ACKNOWLEDGMENTS

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