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Finite Difference/Spectral Approximations for the Fractional Cable Equation *

Yumin Lin† Xianjuan Li‡ Chuanju Xu§

Abstract

In this paper, we consider the numerical solution of the fractional Cable equation, which is a generalization of the classical Cable equation by taking into account the anomalous diffusion in the movement of the ions in neuronal system. A schema combining a finite difference approach in the time direction and a spectral method in the space direction is proposed and analyzed. The main contribution of this work is three-fold: 1) We construct a finite difference/Legendre spectral schema for discretization of the fractional Cable equation. 2) We give a detailed analysis of the proposed schema by providing some stability and error estimates. Based on this analysis, the convergence of the method is rigourously established. We prove that the overall schema is unconditionally stable, and the numerical solution converges to the exact one with order $O(\triangle t^{2-\max\{\alpha,\beta\}})$, where $\triangle t$ is the time step size, α and β are two different exponents between 0 and 1 involved in the fractional derivatives. 3) Finally, some numerical experiments are carried out to support the theoretical claims.

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Key words: Fractional Cable equation, numerical solution, stability, convergence.

1 Introduction

Due to its significant deviation from the dynamics of Brownian motion, the anomalous diffusion in biological systems can not be adequately described by the traditional Nernst-Planck equation or its simplification, the Cable equation. Very recently, a modified Cable equation was introduced for modeling the anomalous diffusion in spiny neuronal dendrites [2]. The resulting governing equation, the so-called fractional Cable equation, is similar to the traditional Cable equation except that the order of derivative with respect to the space and/or time is fractional.

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†School of Mathematical Sciences, Xiamen University, 361005 Xiamen, China. The research of this author was partially supported by Fujian NSF under Grant S0750017.

‡School of Mathematical Sciences, Fuzhou University, 350200 Fuzhou, China.

§School of Mathematical Sciences, Xiamen University, 361005 Xiamen, China. http://lsec.cc.ac.cn/~cjxu.

The goal of this paper is to address such an equation, and to design efficient numerical schemes for its numerical solution. Due to the memory feature of the fractional equation, it is very desirable to use high order methods for efficient computations of the numerical solution. In this work, we aim at developing and analyzing a finite difference schema in time and Legendre spectral methods in space for the fractional Cable equation.

Note that some similar investigations have been made for the time fractional diffusion equation. For example, Langlands and Henry [3] considered an implicit numerical schema for fractional diffusion equation in which the backward Euler approximation is used to discretize the first order time derivative and the L1 schema is used to approximate the fractional order time derivative. Lin and Xu [4] proposed a finite difference schema in time and Legendre collocation spectral method in space for the time fractional diffusion equation.

This work follows the idea proposed in [4] in an attempt to generalize the mixed finite difference/Legendre spectral method in [4] to the fractional Cable equation. In particular, an improved technique, as compared to the one used in [4], for the proof of the time error estimate is provided. This new technique allows to obtain a detailed dependence of the constant appeared in front of the convergence rate $\triangle t^{2-\max\{\alpha,\beta\}}$. The outline of this paper is as follows. In the next section we construct our finite difference method for the Cable equation. A detailed error analysis is carried out to derive the error estimate for the numerical solution. Some numerical results are presented in section 3 which support the theoretical statement.

2 Discretization in time: a finite difference schema

2.1 Cable equation

We consider the initial boundary value problem of the fractional Cable equation. Let $\Lambda = (-1, 1)$ be the space domain, $I = (0, T]$ be the time domain. We consider the fractional Cable equation:

$$\partial_t u = {}^R D_t^\beta \partial_x^2 u - \mu {}^R D_t^\alpha u, \ \forall (x,t) \in \Lambda \times I, \tag{2.1}$$

subject to the initial condition:

$$u(x, 0) = u_0(x), \ \forall x \in \Lambda, \tag{2.2}$$

and the boundary condition:

$$u(-1, t) = u(1, t) = 0, \ \forall t \in I, \tag{2.3}$$

where $0 < \alpha < 1, \ 0 < \beta < 1$. ${}^R D_t^\gamma$, with $\gamma = \alpha$ or β, denotes Riemann-Liouville fractional derivative of γ-order with respect to variable t, defined by

$${}^R D_t^\beta v(x,t) = \frac{1}{\Gamma(1-\beta)} \partial_t \int_0^t \frac{v(x,\tau)}{(t-\tau)^\beta} d\tau, \quad 0 < \beta < 1. \tag{2.4}$$

In order to follow the construction idea used in our previous paper [4], we will use the Caputo fractional derivative instead of the Riemann definition. To this end, we recall the following well-known relation, see e.g. [5]: For $0 < \gamma < 1$, if $v(t)$ has the integrable first order derivative in $[0, T]$, then

$$ {}^{R}D_t^{\gamma} v(t) = D_t^{\gamma} v(t) + \frac{v(0)t^{-\gamma}}{\Gamma(1-\gamma)}, \tag{2.5} $$

where D_t^{γ} denotes the Caputo fractional derivative of γ-order, defined by:

$$ D_t^{\gamma} v(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial_\tau v(\tau)}{(t-\tau)^{\gamma}} d\tau. $$

Then the fractional Cable equation (2.1) can be transformed under the form of Caputo definition:

$$ \partial_t u = D_t^{\beta} \partial_x^2 u - \mu D_t^{\alpha} u + \frac{1}{\Gamma(1-\beta)t^{\beta}} \partial_x^2 u(x,0) - \frac{\mu}{\Gamma(1-\alpha)t^{\alpha}} u(x,0). \tag{2.6} $$

In the next section, we are going to construct and analyze a finite difference schema for the time discretization of the above equation.

2.2 Construction of the schema

First, we introduce a finite difference schema to discretize the time fractional derivative. For a given integer $K > 0$, let $t_k = k\triangle t$, $k = 0, 1, \cdots, K$, where $\triangle t = \frac{T}{K}$ is the time step. By using the Taylor formula with the intergral remainder:

$$ f(t) = f(s) + \partial_t f(s)(t-s) + \int_s^t \partial_\tau^2 f(\tau)(t-\tau) d\tau, \ \forall t, s \in I $$

to the function $u(\cdot, t)$ at $t = t_j$ and $t = t_{j+1}$ respectively, we obtain

$$ \begin{aligned} \partial_s u(x,s) \quad = \quad & \frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} \\ & - \frac{1}{\Delta t} \int_s^{t_{j+1}} \partial_\tau^2 u(x,\tau)(t_{j+1}-\tau) d\tau + \frac{1}{\Delta t} \int_s^{t_j} \partial_\tau^2 u(x,\tau)(t_j - \tau) d\tau. \end{aligned} $$

Thus for all $0 \le k \le K-1$, we have

$$ \begin{aligned} D_t^{\alpha} u(x, t_{k+1}) \quad & = \quad \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \partial_s u(x,s) \frac{ds}{(t_{k+1}-s)^{\alpha}} \\ & = \quad \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} a_j \frac{u(x,t_{k+1-j}) - u(x,t_{k-j})}{\triangle t^{\alpha}} + r_{\alpha}^{k+1}, \end{aligned} \tag{2.7} $$

where

$$ a_j = (j+1)^{1-\alpha} - j^{1-\alpha}, \tag{2.8} $$

and

$$ r_{\alpha}^{k+1} \quad = \quad \frac{1}{\Gamma(2-\alpha)\Delta t} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \partial_\tau u(x,\tau) R_j^{k+1}(\tau) d\tau. \tag{2.9} $$

In the last equality of the above derivation we have used the notation $R_j^{k+1}(\tau)$ to denote

$$R_j^{k+1}(\tau) := (t_{k+1}-\tau)^{1-\alpha}\Delta t - (t_{j+1}-\tau)(t_{k+1}-t_j)^{1-\alpha} + (t_j-\tau)(t_{k+1}-t_{j+1})^{1-\alpha}.$$

It can be proved that

$$R_j^{k+1}(\tau) \geq 0, \quad \text{for all } \tau \in [t_j, t_{j+1}].$$

Thus from equation (2.9), we obtain

$$r_\alpha^{k+1} \quad \leq \quad \frac{M}{\Gamma(2-\alpha)\Delta t}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} R_j^{k+1}(\tau)d\tau,$$

where $M = \max\limits_{x\in\Lambda,\tau\in I} \partial_\tau^2 u(x,\tau)$. Furthermore, a careful analysis shows that

$$r_\alpha^{k+1} \quad \leq \quad c\triangle t^{2-\alpha}, \tag{2.10}$$

where c depends only on M, a constant measuring $\partial_t^2 u$.

We can derive an expression similar to (2.7) for the fractional derivative term of order β in (2.6):

$$D_t^\beta \partial_x^2 u(x,t_{k+1}) = \frac{1}{\Gamma(2-\beta)}\sum_{j=0}^{k}\frac{b_j}{\triangle t^\beta}\left(\partial_x^2 u(x,t_{k+1-j}) - \partial_x^2 u(x,t_{k-j})\right) + r_\beta^{k+1}, \tag{2.11}$$

where

$$b_j = (j+1)^{1-\beta} - j^{1-\beta}, \quad r_\beta^{k+1} \leq c\Delta t^{2-\beta}. \tag{2.12}$$

For the discretization of the first order time derivative $\partial_t u$, we use the following development: for $k \geq 1$,

$$\partial_t u(x,t_{k+1}) = \frac{3u(x,t_{k+1}) - 4u(x,t_k) + u(x,t_{k-1})}{2\Delta t} + O(\Delta t^2), \tag{2.13}$$

For a mesh function $\{f^k\}_{k=0}^K$, we define the fractional difference operators L_t^α by

$$L_t^\alpha f^{k+1} = \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k} a_j \frac{f^{k+1-j} - f^{k-j}}{\triangle t^\alpha}, \; k \geq 0, \tag{2.14}$$

and L_t^β by

$$L_t^\beta f^{k+1} = \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k} b_j \frac{f^{k+1-j} - f^{k-j}}{\triangle t^\beta}, \; k \geq 0, \tag{2.15}$$

We also define the difference operators L_t^1 by

$$L_t^1 f^{k+1} \quad = \quad \frac{3f^{k+1} - 4f^k + f^{k-1}}{2\Delta t}, \; k \geq 1. \tag{2.16}$$

Then by combining (2.6), (2.7), (2.11), and (2.13), we have

$$
\begin{aligned}
(2.17)\qquad & L_t^1 u(x,t_{k+1}) - r^{k+1} \\
= \; & -\mu L_t^{\alpha} u(x,t_{k+1}) + L_t^{\beta}\partial_x^2 u(x,t_{k+1}) - \mu r_{\alpha}^{k+1} + r_{\beta}^{k+1} \\
& + \frac{1}{\Gamma(1-\beta)(k+1)^{\beta}\Delta t^{\beta}}\partial_x^2 u(x,0) - \frac{\mu u(x,0)}{\Gamma(1-\alpha)(k+1)^{\alpha}\Delta t^{\alpha}}, \quad k \geq 1,
\end{aligned}
$$

where, according to (2.13), $r^{k+1} = O(\Delta t^2)$.

The above expression motivates us to consider the following finite difference schema for the time discretization of (2.13):

$$
\begin{aligned}
(2.18)\quad L_t^1 u^{k+1} \; = \; & -\mu L_t^{\alpha} u^{k+1} + L_t^{\beta}\partial_x^2 u^{k+1} \\
& + \frac{1}{\Gamma(1-\beta)(k+1)^{\beta}\Delta t^{\beta}}\partial_x^2 u^0 - \frac{\mu u^0}{\Gamma(1-\alpha)(k+1)^{\alpha}\Delta t^{\alpha}}, \quad k \geq 1.
\end{aligned}
$$

In (2.18), u^k, a simplified notation of $u^k(x)$, is an approximation of $u(x,t_k)$. Formally, (2.18) is a schema with the truncation error $r^{k+1}+r_{\alpha}^{k+1}+r_{\beta}^{k+1}$, stemming respectively from the discretizations of the first order time derivative and the time fractional derivatives of orders α and β.

In details, the schema (2.18) reads

$$
\begin{aligned}
(2.19)\qquad & \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} \\
= \; & \frac{-\mu}{\Gamma(2-\alpha)\Delta t^{\alpha}}\left(u^{k+1} - \sum_{j=0}^{k-1}(a_j - a_{j+1})u^{k-j} - a_k u^0\right) \\
& + \frac{1}{\Gamma(2-\beta)\Delta t^{\beta}}\left(\partial_x^2 u^{k+1} - \sum_{j=0}^{k-1}(b_j - b_{j+1})\partial_x^2 u^{k-j} - b_k \partial_x^2 u^0\right) \\
& + \frac{1}{\Gamma(1-\beta)(k+1)^{\beta}\Delta t^{\beta}}\partial_x^2 u^0 - \frac{\mu u^0}{\Gamma(1-\alpha)(k+1)^{\alpha}\Delta t^{\alpha}}, \quad k \geq 1.
\end{aligned}
$$

2.3 Time error analysis

In this subsection, we aim at carrying out a rigorous error analysis for the time schema (2.19). The error analysis is based on the weak formulation of the related problems. The inner products of $L^2(\Lambda)$ and $H^1(\Lambda)$ are defined respectively by

$$
(2.20)\qquad (u,v) = \int_{\Lambda} uv \; dx, \quad (u,v)_1 = (u,v) + \frac{\tilde{\alpha}}{2}(u,v) + \frac{\tilde{\beta}}{2}(\partial_x u, \partial_x v),
$$

where

$$
(2.21)\qquad \tilde{\alpha} = \frac{4\mu\Delta t}{\Gamma(2-\alpha)\Delta t^{\alpha}}, \quad \tilde{\beta} = \frac{4\Delta t}{\Gamma(2-\beta)\Delta t^{\beta}}.
$$

The norms of $L^2(\Lambda)$ and $H^1(\Lambda)$ are defined by

$$
(2.22)\qquad \|v\|_0 = (v,v)^{1/2}, \quad \|v\|_1 = (v,v)_1^{1/2}.
$$

Here we have used a H^1-norm different from the standard one. We will see that the H^1-norm defined here is more convenient than the standard norm for the error analysis, although the two norms are equivalent for fixed $\mu, \Delta t, \alpha$, and β.

We now consider the weak formulation of the equation (2.19) subject to the homogeneous boundary condition: find $u^{k+1} \in H_0^1(\Lambda)$, such that for all $v \in H_0^1(\Lambda)$,

$$\begin{aligned}
(2.23)\qquad & \left(\frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t}, v\right) \\
= \; & -\frac{\mu}{\Gamma(2-\alpha)\Delta t^\alpha}\Big((u^{k+1}, v) - \sum_{j=0}^{k-1}(a_j - a_{j+1})(u^{k-j}, v) - a_k(u^0, v)\Big) \\
& -\frac{1}{\Gamma(2-\beta)\Delta t^\beta}\Big((\partial_x u^{k+1}, \partial_x v) - \sum_{j=0}^{k-1}(b_j - b_{j+1})(\partial_x u^{k-j}, \partial_x v) - b_k(\partial_x u^0, \partial_x v)\Big) \\
& -\frac{1}{\Gamma(1-\beta)(k+1)^\beta \Delta t^\beta}(\partial_x u^0, \partial_x v) - \frac{\mu}{\Gamma(1-\alpha)(k+1)^\alpha \Delta t^\alpha}(u^0, v), \quad k \geq 1.
\end{aligned}$$

For the sake of simplification, let's introduce the notations:

$$(2.24)\qquad \tilde{\alpha}_{k+1} = \frac{4\mu\Delta t}{\Gamma(1-\alpha)(k+1)^\alpha \Delta t^\alpha}, \quad \tilde{\beta}_{k+1} = \frac{4\Delta t}{\Gamma(1-\beta)(k+1)^\beta \Delta t^\beta}.$$

By using the notations (2.21) and (2.24), the schema (2.23) becomes

$$\begin{aligned}
(2.25)\qquad & 2(3u^{k+1} - 4u^k + u^{k-1}, v) \\
= \; & -\tilde{\alpha}\Big((u^{k+1}, v) - \sum_{j=0}^{k-1}(a_j - a_{j+1})(u^{k-j}, v) - a_k(u^0, v)\Big) \\
& -\tilde{\beta}\Big((\partial_x u^{k+1}, \partial_x v) - \sum_{j=0}^{k-1}(b_j - b_{j+1})(\partial_x u^{k-j}, \partial_x v) - b_k(\partial_x u^0, \partial_x v)\Big) \\
& -\tilde{\beta}_{k+1}(\partial_x u^0, \partial_x v) - \tilde{\alpha}_{k+1}(u^0, v), \; k \geq 1.
\end{aligned}$$

The stability and error estimate are given in the following theorems.

Theorem 2.1. *The semi-discretized problem (2.25) is unconditionally stable in the sense that for all $\triangle t > 0$, it holds*

$$(2.26)\qquad E^{k+1} \leq E^k, \quad k = 1, \cdots, K-1,$$

where $E^k = \|u^k\|_0^2 + \|2u^k - u^{k-1}\|_0^2 + \frac{\tilde{\alpha}}{2}\sum_{j=0}^k a_j\|u^{k-j}\|_0^2 + \frac{\tilde{\beta}}{2}\sum_{j=0}^k b_j\|\partial_x u^{k-j}\|_0^2, \quad k \geq 1.$

Proof. Omitted. □

Theorem 2.2. *Let u be the solution of the continuous problem* (2.1)-(2.3), $\{u^k\}_{k=0}^K$ *be the time-discrete solution of* (2.18). *Then*

$$(2.27)\qquad \|u(x, t_k) - u^k\|_1 \leq cT^\alpha \Delta t^{\min(2-\alpha, 2-\beta)}, \; k \geq 1,$$

where c is independent of T and $\triangle t$.

Proof. Omitted. □

3 Numerical validation

The semi-discretized problem (2.25) is furthermore discretized in space by using a standard spectral method as described in [4]. We omit the details of the description due to the length limit of the paper. We consider the Cable equation with the exact solution $u(x,t) = t^2 \sin(2\pi x)$ for a suitable forcing term. All the numerical results reported in the figures below have been evaluated at $T = 1$.

We fix $N = 16$, a value large enough such that the space discretization error is negligible as compared with the time error. In figures 1, we plot the errors in the L^2 and H^1 semi norms as a function of the time step sizes for two different sets of α, β . A logarithmic scale has been used for both $\triangle t-$axis and error-axis in these figures. As expected, the finite difference schema yields a fractional temporal approximation order $\min\{2-\alpha, 2-\beta\}$, that is, the slopes of the error curves in these $\log - \log$ plots are respectively 1.4 for $\alpha = 0.1, \beta = 0.6$ and 1.1 for $\alpha = 0.9, \beta = 0.1$.

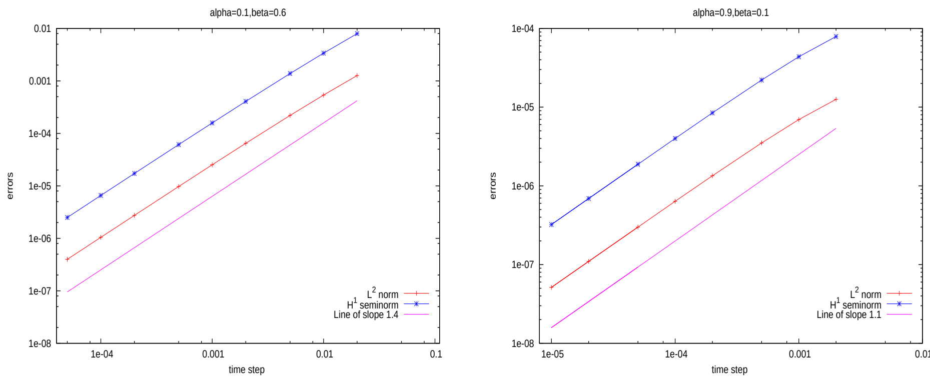

Figure 1: Errors as functions of $\triangle t$ for $\alpha = 0.1, \beta = 0.6$ (left) and $\alpha = 0.9, \beta = 0.1$ (right).

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