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Numerical simulations of techniques related to utility function and price elasticity estimators.

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Abstract: In many economic models the utility function chosen is based on preconceived ideas of the economic state. In order for a utility function to be fit to raw demand data an assumption is made (amongst others) that a preference relation holds within a cycle of data points (Eberhard et al. (2009)). In this paper we assume that errors have occurred in the data collection process which have somewhat corrupted the quantity consumed for the particular commodity price. This results in inconsistences in the preference relation and infeasibility of Afriat inequalities (Eberhard et al. (2009)). We introduce a method that allows the data to shift in order for a Generalised Axiom of Revealed Preference (GARP) to be satisfied by the data enabling a utility to be fitted. This technique is described in section 1. The commonly used Cobb-Douglas utility is defined as

$$U(x_i, ..., x_L) = \prod_{i=1}^{L} x_i^{\alpha_i}, \tag{1}$$

where $\sum_{i=1}^{L} \alpha_i = 1$. This function represents the demand of commodities with respect to commodity costs and household income. Here α represents the commodity share of good i in the total household expenditure. Upon solving the utility maximisation problem the consumer spends the entire household budget $\langle x_i, p_i \rangle = y$, and the demand is given as

$$x_i = \frac{\alpha_i y}{p_i}. \tag{2}$$

We run simulations with generated Cobb-Douglas type price-demand data to compare the fit of an "Afriat" type utility. Errors are included in the Cobb-Douglas data to simulate the corruption of GARP and to test the robustness of the error shifting least squares program. Calculating price elasticities of demand is an important part of an economic model, as it quantifies the susceptibility to change in quantity consumed for an associated change in commodity price. The Hicks-Slutsky partition (Dixon et al. (1980)) describes the changes in demand given by a price change in commodity in conjunction with the associated change in demand given by the consumers change in income. We fit a utility to simulated Cobb-Douglas demand to compare our calculated elasticities (constrained to be consistent with the Hicks-Slutsky Partition) with the known Cobb-Douglas elasticities. Cobb-Douglas price elasticity of demand is unitary as demonstrated by the demand relation (2), therefore a 1% increase/decrease in price will lead to a 1% decrease/increase in demand. Cross-price elasticity for this case is zero, since the demand is directly proportional to it's own price. We investigate and compare the elasticities generated by our method. An example of commodities that are considered to be perfect substitutes is tea and coffee. Using the technique described in section 2 we run a simulation of calculated elasticities with Cobb-Douglas simulated data and also calculate elasticities of real price-quantity data from the ABS to determine the substitutability of tea-coffee.

Keywords: *Economic modelling, utility function, revealed preferences, price elasticity of demand*

1 CONSUMER DEMAND AND THE REVEALED PREFERENCE

A preference relation $x\mathcal{R}y$ states that x is a revealed preference to y, (Houthakker, (1950)) and more recently (Fostel et al. (2004)). We observe data $x_i \in X_{\mathcal{R}}(p_i)$, that is x_i is observed in the demand relation at price i. In order for a preference relation to exist we should have for all cycles of data of length m

$$X = \{(x_i, p_i) \mid i = 1, \ldots, m\} \, (\text{with } x_1 = x_{m+1})$$

$$\text{that all } \langle p_{i+1}, x_{i+1} \rangle - \langle p_{i+1}, x_i \rangle \geq 0 \qquad \Longrightarrow \qquad \langle p_{i+1}, x_{i+1} \rangle - \langle p_{i+1}, x_i \rangle = 0.$$

We note that $x_{i+1} \succeq_{\mathcal{R}} x_i$, x_{i+1} at price p_{i+1} is a revealed preference to x_i since x_i was in budget but not chosen. i.e $x_{i+1} \in X(p_{i+1})$, this can be used to sort the demands in terms of preference levels ensuring that there are no contradictions.

When obtaining a finite sample of consumer demand data, the demand relation may have inconsistencies due to the demographic of the data gathering process. The set of data samples of demand x_i and price p_i for $i \in I$ gives a finite data set $\{(x_i, p_i)\}_{i \in I}$, $I = \{1 \ldots k\}$ of observed commodities $x_i \in \mathbb{R}^L$. We assume the observed data is of the form $x_i = \bar{x_i} + \bar{s_i}$, where the correct data $\bar{x_i}$ is corrupted by an "unseen" error $\bar{s_i}$. Hence the data pairs do not satisfy the General Axiom of Revealed Preferences (GARP)(Eberhard et al. (2009)).

We now introduce an error s_i so that the observed demand x_i can vary in order for the data to satisfy GARP. This leads to a quadratic least squares minimisation problem,

$$\begin{aligned} \min_{(\phi,\lambda,s)} & \sum_{i\in I} ||s_i||^2 + \sum_{i \in I} \lambda_i \\ \text{subject to} & \\ & x_i + s_i \geq 0, \\ & \langle s_i, p_i \rangle = 0, \\ & \lambda \geq 1, \\ & \phi_j - \phi_i \leq \lambda_i \left[\langle p_i, x_j - x_i \rangle + \langle p_i, s_j - s_i \rangle \right] \text{for } i, j \in I. \end{aligned} \tag{LS-QP}$$

Provided a feasible solution exists then the utility is given by

$$u^{-}(x) := \min_{i \in I} \left\{ \phi_i + \lambda_i p_i^T (x - x_i) \right\} \tag{3}$$

The idea is to try to estimate the unknown errors $\bar{s_i}$ using s_i. How well can (LS-QP) perform in cases where the data contains large errors? Does (LS-QP) shift the s_i's sufficiently to return to the original demand $\bar{x_i}$? We answer these questions by performing a sensitivity analysis on the slack values s_i by introducing errors to data that initially satisfied GARP. By randomly generating price data of the form $(\bar{x_i} + \bar{s_i}, p_i)$ from a Cobb-Douglas utility we run (LS-QP) and compare the shifts in the slacks s_i with the introduced errors $\bar{s_i}$. We solve (LS-QP) $N = 30$ times for a size m sample to find an average error and grand error of the data respectively

$$a_n = \sum_{i=1}^{m} \|s_i - \bar{s_i}\| \quad \text{for } n = 1, \ldots, N \qquad \text{and } err = \frac{1}{N} \sum_{n=1}^{N} a_n \tag{4}$$

of the slacks. The 95% confidence interval is displayed below.

Table 1: 95% Confidence Interval of Slack Errors

	Sample Size		
Error	5	10	30
0.001	$(8.238700 \times 10^{-9}, 1.43521 \times 10^{-7})$	$(1.52842 \times 10^{-5}, 4.89921 \times 10^{-5})$	$(1.74478 \times 10^{-5}, 4.14146 \times 10^{-5})$
0.01	$(-5.162467 \times 10^{-6}, 1.22444 \times 10^{-5})$	$(-1.59320 \times 10^{-3}, 4.53031 \times 10^{-3})$	$(-2.18439 \times 10^{-3}, 5.58878 \times 10^{-3})$
0.1	$(-6.06453 \times 10^{-4}, 1.11954 \times 10^{-3})$	$(-2.78953 \times 10^{-1}, 5.48663 \times 10^{-1})$	$(-3.29819 \times 10^{-1}, 6.71475 \times 10^{-1})$

From Table 1 we can see that for small errors in the data $\bar{s} = 0.001$ (LS-QP) does not shift the slacks s_i. We conjecture that GARP is already satisfied being insensitive to small changes. The range of the confidence interval is significantly small although positive since the introduce error is negligible. It is pleasing to note that (LS-QP) is working surprisingly well for smaller sample sizes and correcting the larger introduced errors.

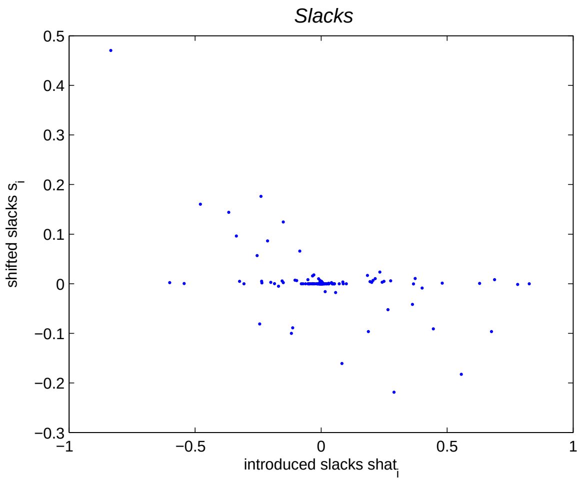

Figure 1: Introduced errors compared with data shifting slacks

Figure 1 shows the error and shifted error clustered around zero, this demonstrates the ability of the slack variables s_i to recognise the size of error $\bar{s}_i$ in order to shift the corresponding data to eliminate a large portion of the error.

2 PRICE ELASTICITIES OF DEMAND

The Hicks-Slutsky Partition describes the change in demand of a commodity with respect to the change in the price. Price elasticity of demand is defined in two components. The first component is often called the substitution (or compensated) effect. Here the consumer is able to enjoy the same (fixed) level of utility on an unconstrained budget. The consumer is able to alter their demand for commodities based on the changing prices which in turn will also alter their budget.
The second component is known as the income (uncompensated) effect. Here the consumer's budget can increase/decrease without an associated change in commodity prices. The consumer will experience a different level of utility based on the change in quantity they are able to consume. This is also referred to as uncompensated elasticity as a decrease in the consumers budget will restrict the consumers demand. Mathematically, the Hicks-Slutsky Partition describes the relationship between these elasticities.

$$\frac{\partial x_i}{\partial p_j} = \left(\frac{\partial x_i}{\partial p_j}\right)_{dU=0} - x_j \frac{\partial x_i}{\partial y}, \tag{5}$$

or equivalently

$$e_{ij} = e^c_{ij} - \alpha_j E_i. \tag{6}$$

The share value $\alpha_i = \frac{x_i p_i}{y}$, is the amount a consumer spends on a commodity in relation to their entire budget, the first term in (5) is the change in demand given by a change in price whilst holding the consumer to a fixed utility level. The second term in (5) is the uncompensated (Engel) elasticity which is the consumers elasticity of change in demand with respect to a change in household income. Since the utility has been fitted using parameters satisfying the "Afriat" inequalities we obtain a polyhedral function thus allowing us to define the elasticities via a linear program (LP).

2.1 Calculating Compensated Price Elasticity of Demand

We use the following approximation

$$e^c_{ij} = \left(\frac{\partial x_i}{\partial p_j}\right)_{dU=0} \approx \frac{p_j}{x_i}\frac{\Delta x_i}{\Delta p_j} \tag{7}$$

To calculate the associated change in demand we require the utility to remain fixed whilst the commodity prices are allowed to vary. The optimisation problem can then be posed in the form,

$$\begin{aligned} &\min\langle p, X\rangle \\ &\text{subject to } X \geq 0 \\ &\qquad u(X) \geq u(x_0) \end{aligned} \tag{8}$$

Here p and X denote the (L × 1) price and quantity vectors respectively. For $P_0 = (p_1^*, \ldots, p_L^*)^T$ (the prices at equilibrium) it is expected that $X_0 = (x_1^*, \ldots, x_L^*)^T$. As before the utility is defined by (3),

Problem (8) can now be written as a parametric linear program,

$$\begin{array}{rl} \min & \langle p, X\rangle \\ \text{subject to} & X \geq 0 \\ & \phi_0 \leq \phi_i + \lambda_i\langle p, X - x_i\rangle \quad \forall \quad i = 1, \ldots, k \end{array} \tag{LP(P)}$$

We must solve the $(LP(P_0))$ for $X_0 = (x_1^*, \cdots, x_L^*)^T$ and $P_0 = (p_1^*, \cdots, p_L^*)^T$ and check the sensitivity of this solution X_0 with respect to changes in the price P_0 in the objective function. By increasing/decreasing each commodity price in P_0 we obtain a new optimal solution. This enables us to determine an interval

$$\begin{pmatrix} P_0^- \\ P_0^+ \end{pmatrix} = \begin{pmatrix} p_1^{*-} & p_2^{*-} & \cdots & p_L^{*-} \\ p_1^{*+} & p_2^{*+} & \cdots & p_L^{*+} \end{pmatrix} \tag{9}$$

containing P_0 in which the solution X_0 remains optimal for $(LP(P_0))$.

Begin by decreasing the lower bound by defining $P_j^- = P_0^- - \varepsilon l_j$ where $0 < \varepsilon \ll 1$, and l_j is a $L \times 1$ vector of zeros with 1 in the j^{th} row. The associated demand is calculated by solving for $(LP(P_j^-))$ and compared with the optimal demand. The price change matrix can now be written as,

$$\begin{pmatrix} P_j^- \\ P_j^+ \end{pmatrix} = \begin{pmatrix} P_0^- - l_j\varepsilon \\ P_0^+ + l_j\varepsilon \end{pmatrix} \tag{10}$$

The changes are made for each change in price p_j calculate the change in x_i demand and is recorded as x_{ij}. Once the solution has moved from the optimal demand (given a acceptable tolerance), the new optimal demand for the given price change is recorded and the process repeated for all commodities. The demand change matrix for the lower price and upper price change is stored as:

$$X^- = \begin{pmatrix} x_{11}^- & x_{12}^- & \cdots & x_{1L}^- \\ x_{21}^- & x_{22}^- & \cdots & x_{2L}^- \\ \vdots & \vdots & \ddots & \vdots \\ x_{L1}^- & x_{L2}^- & \cdots & x_{LL}^- \end{pmatrix} X^+ = \begin{pmatrix} x_{11}^+ & x_{12}^+ & \cdots & x_{1L}^+ \\ x_{21}^+ & x_{22}^+ & \cdots & x_{2L}^+ \\ \vdots & \vdots & \ddots & \vdots \\ x_{L1}^+ & p_{L2}^+ & \cdots & x_{LL}^+ \end{pmatrix} \tag{11}$$

Therefore the change in demand and price can be written as

$$\Delta x_i = X_{ij}^+ - X_{ij}^- \qquad \text{and} \quad \Delta p_i = P_j^+ - P_j^- \tag{12}$$

respectively. The compensated price elasticity of demand can be defined as

$$e_{ij}^c = \frac{p_j}{x_i}\frac{\Delta x_i}{\Delta p_j} \approx \frac{P_{0j}}{X_{0i}}\frac{X_{ij}^+ - X_{ij}^-}{P_j^+ - P_j^-} \qquad \forall i = 1, \ldots, L \tag{13}$$

2.2 Calculating Uncompensated Price Elasticity of Demand (Engel Aggregation)

We use the approximation

$$E_i = \frac{y}{x_i}\frac{\partial x_i}{\partial y} \approx \frac{y}{x_i}\frac{\Delta x_i}{\Delta y} \tag{14}$$

The uncompensated elasticity allows the consumer to maximise their utility subject to a budget constraint y whilst holding commodity prices fixed. The utility is still bounded below by the utility levels of the other commodity bundles. Here the consumer can move from each utility curve in order to maximise their utility. In terms of our fitted utility u^- the maximisation problem becomes,

$$\begin{aligned} \max_{(X,z)} \quad & z \\ \text{subject to} \quad & \langle X, p\rangle \leq y \\ & X \geq 0 \\ & z \leq \phi_i + \lambda_i \langle p, X - x_i\rangle \quad \forall \quad i = 1, \dots, k \end{aligned} \tag{LP(Y)}$$

The optimal solution $X_0^E = (x_1^*, \cdots, x_L^*)^T$ is found by solving (LP(Y)) for the given budget y. As any change in budget will lead to a change in quantity demanded we can define the lower change in budget as $Y^- = y - \mu$ and solve $(LP(Y^-))$. Similarly define $Y^+ = y + \mu$ and solve $(LP(Y^+))$. Where $0 < \mu << 1$.
The budget change matrix is then,

$$\begin{pmatrix} Y^- \\ Y^+ \end{pmatrix} = \begin{pmatrix} Y_0 - \mu \\ Y_0 + \mu \end{pmatrix} \tag{15}$$

The demand change matrix for the associated decrease and increase in budget is stored as:

$$X^{E-} = \begin{pmatrix} x_1^- \\ x_2^- \\ \vdots \\ x_L^{E-} \end{pmatrix} \quad \text{and} \quad X^+ = \begin{pmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_L^+ \end{pmatrix} \tag{16}$$

The Engel aggregation (uncompensated elasticity) (14) is now defined as

$$E_i \approx \frac{Y}{x_i} \frac{X_i^{E+} - X_i^{E-}}{Y^+ - Y^-} \qquad \forall i = 1, \dots, L \tag{17}$$

The elasticity (5) is now represented as a linear change,

$$e_{ij} = e_{ij}^c - \alpha_j E_i \approx \frac{P_{0j}}{X_{0i}} \frac{X_{ij}^+ - X_{ij}^-}{P_j^+ - P_j^-} - \alpha_j \frac{Y_0}{X_{0i}^E} \frac{X_i^{E+} - X_i^{E-}}{Y^+ - Y^-} \tag{18}$$

3 APPLICATION TO CALCULATING ELASTICITIES OF A COBB-DOUGLAS UTILITY FUNCTION

To compare the calculated elasticities with the known results of a Cobb-Douglas utility, samples of 2 commodity bundles of size 30 were generated and used to calculate the elasticities as described in section 2. It is expected that the optimal solution to the price minimisation problem $(LP(P_0))$, and utility maximisation problem (LP(Y)) are equal ie. $(X_0 = X_0^E)$ for the optimal price sample P_0 since both $(LP(P_0))$, and (LP(Y)) maximise the consumers utility subject to the budget constraint. As the demand calculated for a Cobb-Douglas type utility is $x_i = \frac{\alpha Y}{p_i}$, an increase in the price of commodity i will decrease the quantity demanded and similarly a price decrease will cause an increase in quantity demanded.
The price and demand data is randomly generated in bundles of 10 as the data follows a normal distribution. By generating smaller bundles the data clusters around the general equilibrium point (x_1^{opt}, x_2^{opt}) giving a smoother approximation around the clustered data and hence providing more information around the optimal point. This can be seen in figure (2). The calculated elasticities are:

$$e = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} -1.1297 & 0.099 \\ -0.0225 & -0.9746 \end{pmatrix} \tag{19}$$

The own-price elasticities represented by e_{11} and e_{22} being ≈ -1 agree with the data being of a Cobb-Douglas type utility as the elasticity of -1 is unitary elastic as a 1% increase in the price of commodity i will lead to a 1% decrease in demand of commodity i. Similarly the Cross-Price elasticities e_{12} and e_{21} being close to zero show that the price change in commodity j does not affect the demand of commodity i.

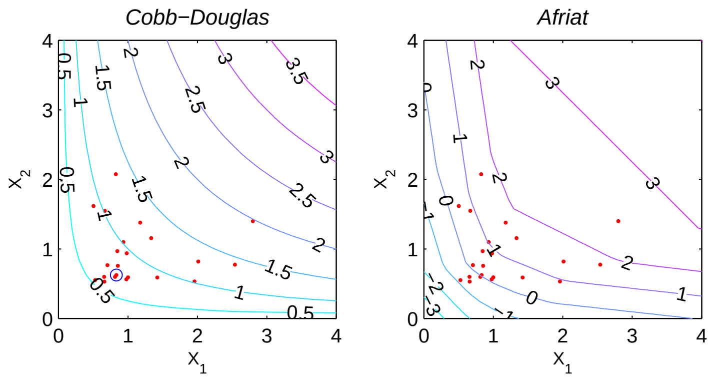

Figure 2: Cobb-Douglas Utility curve and Afriat fit

4 APPLICATION TO CALCULATING ELASTICITIES OF TEA AND COFFEE

Using the Australian Bureau of Statistics (ABS) household expenditure data from tables (2) and (3) respectively, the price and demand are now used as the input into (LS-QP) to fit a utility function to the data. Now using the consumer data and the values calculated for ϕ, λ and s as an input to (LP(P)) and (LP(Y)).

Table 2: Price and Quantity of 180g Bags of Tea for 1993,1999 and 2004

Tea	**Price**			**Quantity**		
Sydney	\$1.85	\$3.27	\$3.75	0.31	0.20	0.19
Melbourne	\$1.83	\$3.38	\$3.45	0.33	0.19	0.20
Brisbane	\$1.64	\$3.47	\$3.52	0.30	0.16	0.19
Adelaide	\$1.59	\$3.25	\$3.44	0.31	0.15	0.18
Perth	\$1.78	\$3.34	\$3.37	0.35	0.20	0.21
Hobart	\$1.99	\$3.99	\$3.77	0.32	0.17	0.21
Darwin	\$1.94	\$3.35	\$3.70	0.22	0.19	0.14
Canberra	\$1.87	\$3.33	\$3.83	0.30	0.23	0.23

Table 3: Price and Quantity of 150g Jar of Coffee for 1993,1999 and 2004

Coffee	**Price**			**Quantity**		
Sydney	\$3.74	\$5.82	\$5.97	0.25	0.23	0.20
Melbourne	\$5.03	\$6.03	\$5.72	0.21	0.26	0.25
Brisbane	\$4.07	\$6.04	\$5.53	0.22	0.20	0.22
Adelaide	\$5.12	\$5.68	\$5.42	0.20	0.27	0.21
Perth	\$4.69	\$6.44	\$6.47	0.19	0.17	0.16
Hobart	\$4.99	\$6.93	\$6.86	0.20	0.18	0.22
Darwin	\$5.56	\$6.32	\$5.80	0.14	0.22	0.23
Canberra	\$4.39	\$5.70	\$6.10	0.25	0.31	0.26

The price input the average of all tea and coffee prices over the three time periods. We normalise the budget to 1 and take the normalised average prices as $(p_1, p_2)_1 = (1.55, 2.94)$. Upon solving (LP(P)) the optimal solution is given as

$$X^{opt} = \begin{pmatrix} x_1^{opt} \\ x_2^{opt} \end{pmatrix} = \begin{pmatrix} 0.2294 \\ 0.2195 \end{pmatrix}. \tag{20}$$

The calculated elasticities of demand are,

$$e = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11}^c - \alpha_1 E_1 & e_{12}^c - \alpha_2 E_1 \\ e_{21}^c - \alpha_1 E_2 & e_{22}^c - \alpha_2 E_2 \end{pmatrix} = \begin{pmatrix} -1.136 & -0.0465 \\ -0.01 & -0.903 \end{pmatrix}. \tag{21}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0.3547 \\ 0.6453 \end{pmatrix}. \tag{22}$$

The elasticities show that own-price elasticities are negative which agrees with the price demand theory that a increase in own price will lead to a decrease in quantity demanded. The values of the own price elasticities $e_{11} = -1.136$ and $e_{22} = -0.903$ demonstrate almost unitary elasticities.

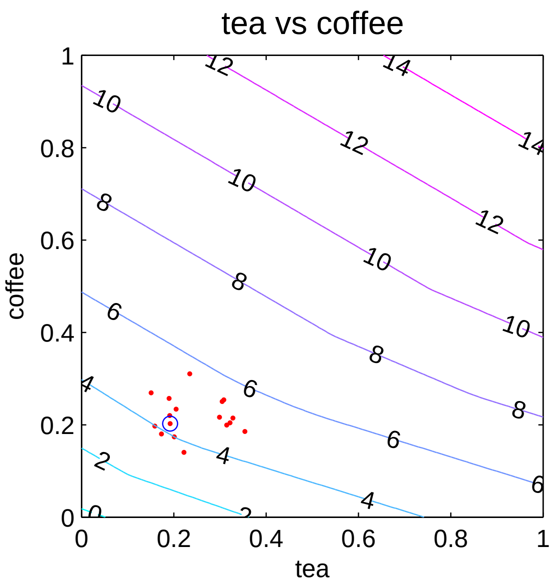

Figure 3: Demand of Tea and Coffee

Figure (3) demonstrates the substitutability of the two commodities. The Afriat Utility has fit parallel lines to the data which are slightly skewed to the right. Tea and coffee are still considered to be perfect substitutes with coffee favoured slightly more than tea as one would give up less coffee to gain more tea.

5 CONCLUSION

The technique used to calculate the utility function based on consumer demand data has proven to be robust even for small data samples. The calculation of elasticities has demonstrated the substitutability of tea and coffee to be as expected. For the generated Cobb-Douglas data the elasticities also agree with the expected elasticities. Provided that the data is well clustered around the optimal solution then an appropriate change in demand is found, hence leading to more accurately calculated elasticities.

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