

Modeling old-age mortality risk for the populations of Australia and New Zealand: an extreme value approach

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Abstract: Old-age mortality for populations of developed countries has been improving rapidly since the 1950s. This phenomenon, which is often referred to as ‘rectangularization’ of mortality, implies an increased survival at advanced ages. At the time of the 2006 Census, there were 3,154 centenarians in Australia. This number is expected to increase to 17,408 by 2028, according to the Australian Bureau of Statistics. With this increase comes different challenges to actuaries, economists and policy planners. A reliable estimate of old-age mortality would definitely help them develop various demographic and financial projections.

Unfortunately, data quality issues have made the modeling of old-age mortality difficult. In more detail, the number of deaths and exposures-to-risk dwindle at advanced ages, leading to large sampling errors and highly volatile crude death rates. Moreover, ages at death are often misreported. This is because today’s centenarians were born more than 100 years ago, when record-keeping was not as exhaustive as it is today. Therefore, we need a method that can extrapolate a survival distribution to extreme ages without requiring accurate mortality data for the centenarian population.

In this paper, we focus on a method called the threshold life table, proposed by Li *et al.* (2008). The threshold life table systematic integrates extreme value theory to the parametric modeling of mortality. More specifically, it uses the asymptotic distribution of the exceedances over threshold to model the survival distribution beyond a particular age. This age, which is known as the threshold age, is chosen to ensure that the tail of the fitted distribution is consistent with the parametric graduation for earlier ages.

We use the threshold life table to model the most recent period (static) mortality rates for the populations of Australia and New Zealand. We observe a good fit to the raw data for both populations. Also, the transition from a parametric graduation to an extreme value extrapolation is fairly smooth, since the estimation method focuses on both the tail and the body of the survival distribution. Further, from the fitted models we can tell in what way the life tables for the two populations should be closed.

We then extend the model to predict the highest attained age, which is commonly referred to as ‘omega’ or ω in the actuarial literature, for the populations of Australia and New Zealand. On the basis of the threshold life table, the central estimates of ω for Australia and New Zealand are 112.20 and 109.43, respectively. Our estimates of ω are reasonably consistent with the validated supercentenarian in these countries.

Keywords: *Extreme value theory, Highest attained age, Old-age mortality*

1. INTRODUCTION

Old-age mortality for populations of developed countries has been improving rapidly since the 1950s. This phenomenon, which is often referred to as ‘rectangularization’ of mortality, implies an increased survival at advanced ages. In Australia, the number of centenarians have increased by 8.5% per year over the past 25 years (Richmond, 2008). At the time of the 2006 Census, there were 3154 centenarians in Australia, 797 men (25%) and 2357 women (75%). This number is expected to increase to 17,408 by 2028, according to the population projections made by the Australian Bureau of Statistics.

With this increase comes different challenges to the public health and pension systems as they try to meet people’s needs. A reliable estimate of mortality at advanced ages would help policy planners better adapt current systems to suit future demands. In the insurance industry, the rapid increase in the number of centenarians may lead to low-frequency, high-severity losses in businesses which involve cash flows that are contingent on survival. Prime examples include life annuities, defined-benefit pension plans and reverse mortgages. Actuaries would need accurate old-age mortality rates to price and reserve these products appropriately.

Unfortunately, data quality issues have made the modeling of old-age mortality difficult. In more detail, the number of deaths and exposures-to-risk dwindle at advanced ages, leading to large sampling errors and highly volatile crude death rates. Moreover, ages at death are often misreported. This is because today’s centenarians were born more than 100 years ago, when record-keeping was not as exhaustive as it is today. Huge effort is required to validate their ages at death (see, e.g., Bernard and Vaupel, 1999; Bourbeau and Desjardins, 2002), and it is often impossible to confirm or disprove the accuracy in the absence of documentary evidence.

Actuaries and demographers have suggested various solutions to this problem. These methods are based on different forms of mathematical extrapolation. For example, Panjer and Russo (1992) and Panjer and Tan (1995) use a cubic polynomial to extend survival distributions beyond age 100; Heligman and Pollard (1980) use a discrete version of the Gompertz law (Gompertz, 1825) to model senescent mortality; Coale and Guo (1989) and Coale and Kisker (1990) suggest a method that is based on the assumption that old-age mortality rates increase at a varying rate; Himes *et al.* (1994) propose a ‘standard’ mortality schedule on which an extrapolation of life tables may be based. A shortcoming of these methods is that there is little statistical justification for the assumed trajectories of old-age mortality.

Recent developments in extreme value theory may offer an alternative solution to the problem. Li *et al.* (2008) propose a method called the threshold life table, which systematic integrates extreme value theory to the parametric modeling of mortality. This method allows us to extrapolate a survival distribution to extreme ages without the need for accurate mortality data at extreme ages, and to determine statistically in what way a life table should be closed. In this paper, we model old-age mortality for the populations of Australia and New Zealand by using the threshold life table. We then extend the model to predict the highest attained age, which is commonly referred to as ‘omega’ or ω in the actuarial literature, for the two populations.

The paper proceeds as follows. Section 2 provides the specification of the threshold life table and describes how the model parameters may be estimated. Section 3 applies the threshold life table to the populations of Australia and New Zealand. Section 4 considers the use of the threshold life table to estimate the highest attained age in Australia and New Zealand. The discussion and conclusion follow in the final section.

2. THE THRESHOLD LIFE TABLE

2.1 Motivation

Let us first take a look at the age patterns of mortality for the populations of Australia and New Zealand. From Figure 1 we observe that, for both age patterns, the gradients beyond age 85 are significantly higher than before. Given this pattern, the modeling of static mortality is usually performed in a piece-wise approach. The first piece is a graduation that is applied to early-age death rates. Such a graduation is intended to smooth out the raggedness in the crude death rates and to summarize of the age patterns by a limited number of parameters. The second piece is an extrapolation of the graduated death rates to advanced ages. There are two key issues that must be addressed in modeling with the piece-wise approach: (1) how should the life tables be closed? (2) at what age should the extrapolation begin?

The threshold life table is also a piece-wise approach. In a threshold life table, death rates at earlier ages are graduated by a parameterized function. The rates are then extended to advanced ages by a statistical distribution

that is justified by extreme value theory. From the parameters of the extreme value distribution, we can tell whether a life table should be closed with $q_\omega = 1$ at a certain limiting age ω . If so, we can also tell from the parameters an estimate of ω . Further, in a threshold life table, the age at which the modeling switches from a parametric graduation to an extreme value distribution is wholly determined by statistical means. Subjective decisions are not required.

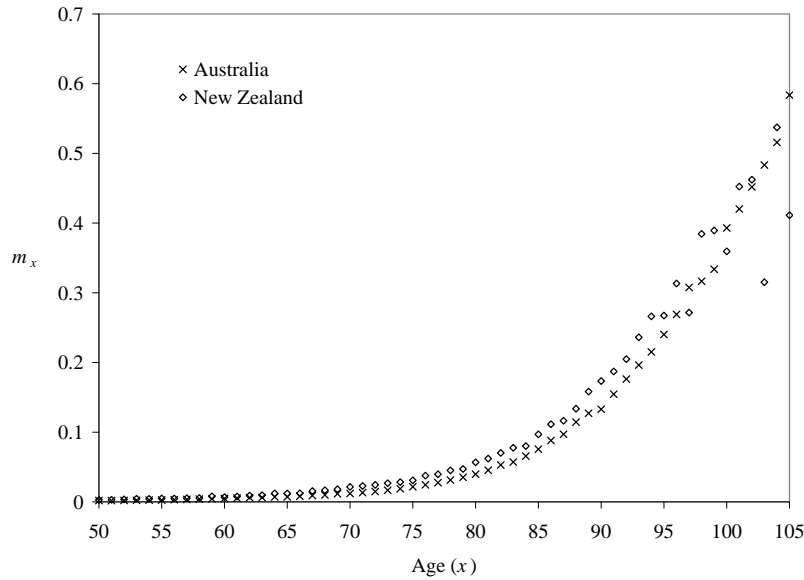


Figure 1. Crude central death rates from age 50, the populations of Australia and New Zealand

2.2 Model Specification

Let us define the following notation:

- X : the age at death random variable, assumed to be continuous.
- $f(x)$: the probability density function of X .
- $F(x)$: the distribution function of X .
- $S(x) = 1 - F(x)$: the survival function.
- $\mu(x) = f(x)/(1 - F(x))$: the force of mortality.
- d_x : the number of deaths between ages x and $x + 1$.
- E_x : the number of exposures-to-risk between ages x and $x + 1$; in practice, E_x is approximated by the mid-year population at age x .
- l_x : the number of survivors to age x .
- $m_x = d_x/E_x$: the central rate of death at age x .
- $q_x = d_x/l_x$: the probability of death between ages x and $x + 1$, conditioning on survival to age x .

Let $Z = \{Y - d | Y > d\}$ be the conditional exceedance of Y over a threshold d . The Balkema-de Haan-Pickands Theorem (Balkema and de Haan, 1994) states that, under certain regulatory conditions, the limiting distribution of Z is a Generalized Pareto distribution as the threshold d approaches the right-hand end support of Y . This important result in extreme value theory provides a theoretical justification of the threshold life table, which is defined as follows:

$$F(x) = \begin{cases} 1 - \exp\left(-\frac{B}{\ln C}(C^x - 1)\right), & x \leq N \\ 1 - p\left(1 + \gamma\left(\frac{x-N}{\theta}\right)\right)^{-1/\gamma}, & x > N. \end{cases}$$

where $p = S(N)$, and N is known as the *threshold age*. In other words, we assume that the survival distribution is Gompertzian before the threshold age, and the exceedances over the threshold age follow a Generalized Pareto distribution, according to the Balkema-de Haan-Pickands Theorem. To ensure that F is a proper distribution function, we require $B > 0$, $C > 0$, and $\theta > 0$. Such a specification ensures that F is continuous at the threshold age. However, it does not guarantee that F is smooth during the transition from graduation to extrapolation. To achieve smoothness, we require a careful choice of the threshold age.

In many applications of Generalized Pareto distributions, the threshold d can be determined by the Hill estimator (Hill, 1975; Mason, 1982), which is defined as follows:

$$\hat{\alpha}(k) = \left(\frac{1}{k} \sum_{i=1}^k \ln \frac{X_{i,n}}{X_{k+1,n}} \right)^{-1},$$

where $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ is a random sample of size n arranged from largest to smallest. We may take $X_{j+1,n}$ as an estimate of the threshold d if the plot of $\hat{\alpha}(k)$ becomes flat when k is greater than j .

Unfortunately, for two reasons, the Hill estimator cannot be used for selecting the threshold age. First, the data we use in this study are censored, by which we mean we know only the number of deaths in different age intervals but not the precise age at death for each individual. However, the Hill estimator defined above does not work when there exists censoring. Second, the Hill estimator focuses only on the tail but ignores the body of a distribution. As a result, using the Hill estimator will not produce a model that is suitable for the entire life-span. An alternative method for choosing the threshold age is therefore required.

2.3 Maximum Likelihood Estimation

Li *et al.* (2008) propose the method of maximum likelihood for the threshold life table model estimation. In describing the method, we assume that we are provided with period (static) mortality data for ages 65 to 99 and the open age group 100 & above.

The likelihood function for the threshold life table can be written as

$$L(B, C, \gamma, \theta; N) = \left(\prod_{x=65}^{99} \left(\frac{S(x) - S(x+1)}{S(65)} \right)^{d_x} \right) \times \left(\frac{S(100)}{S(65)} \right)^{l_{100}}.$$

We can easily show that the logarithm of $L(B, C, \gamma, \theta; N)$ can be written as the sum of two components, $l_1(B, C; N) + l_2(\gamma, \theta; N)$, where

$$l_1(B, C; N) = \sum_{x=65}^{N-1} (d_x \ln(S(x) - S(x+1))) + l_N \ln(S(N)) - l_{65} \ln(s(65)),$$

where $S(x) = \exp\left(\frac{-B}{\ln C}(C^x - 1)\right)$, and

$$l_2(\gamma, \theta; N) = \sum_{x=N}^{99} \left(d_x \ln \left(\frac{S(x)}{S(N)} - \frac{S(x+1)}{S(N)} \right) \right) + l_{100} \ln \left(\frac{S(100)}{S(N)} \right),$$

where $\frac{S(x)}{S(N)} = \left(1 + \gamma \left(\frac{x-N}{\theta}\right)\right)^{-1/\gamma}$.

This above implies that for a given N , the Gompertz part and the Generalized Pareto part can be estimated separately by maximizing l_1 and l_2 , respectively. We then choose N by maximizing the following profile log-likelihood function:

$$l_p(N) = l \left(\hat{B}(N), \hat{C}(N), \hat{\gamma}(N), \hat{\theta}(N); N \right),$$

where $l = \ln(L)$; $\hat{B}(N)$, $\hat{C}(N)$, $\hat{\gamma}(N)$, and $\hat{\theta}(N)$ are the maximum likelihood estimates of B , C , γ , and θ for fixed N , respectively. Summing up, the threshold age and model parameters can be estimated by the algorithm below:

1. for $N = 98$,
 - (a) find the values of B and C that maximize l_1 ;

- (b) find the values of γ and θ that maximize l_2 ;
 - (c) compute the value of the profile log-likelihood, l_p ;
2. repeat Step (1) for $N = 97, 96, \dots, 66$;
 3. find the value of N that yields the maximum profile log-likelihood.

The value of N obtained in step (3) is the optimal threshold age. The maximum likelihood estimates of B, C, γ and θ on the basis of the optimal threshold age are the optimal model parameter values.

3. DATA AND EMPIRICAL RESULTS

We use threshold life tables to model the most recent period (static) mortality rates for the populations of Australia and New Zealand. The data are provided by the Human Mortality Database (2008).

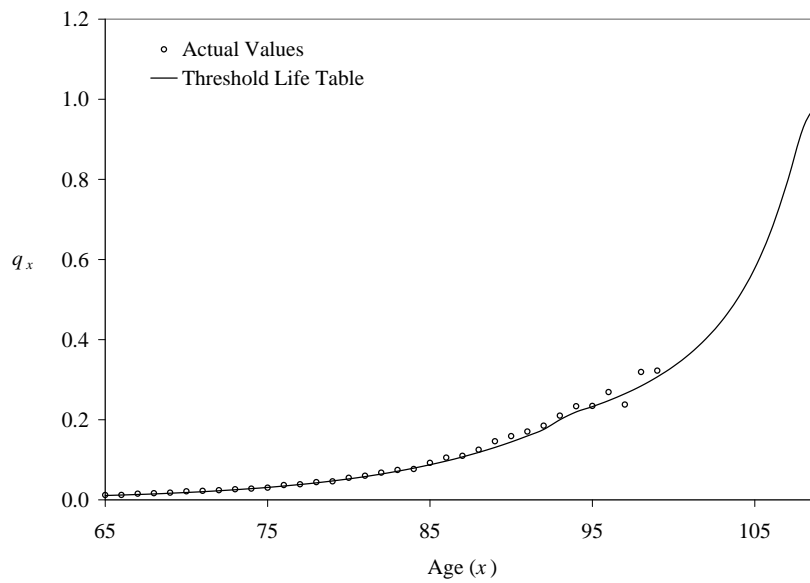


Figure 2. Estimated threshold life table for the population of Australia ($N=96$)

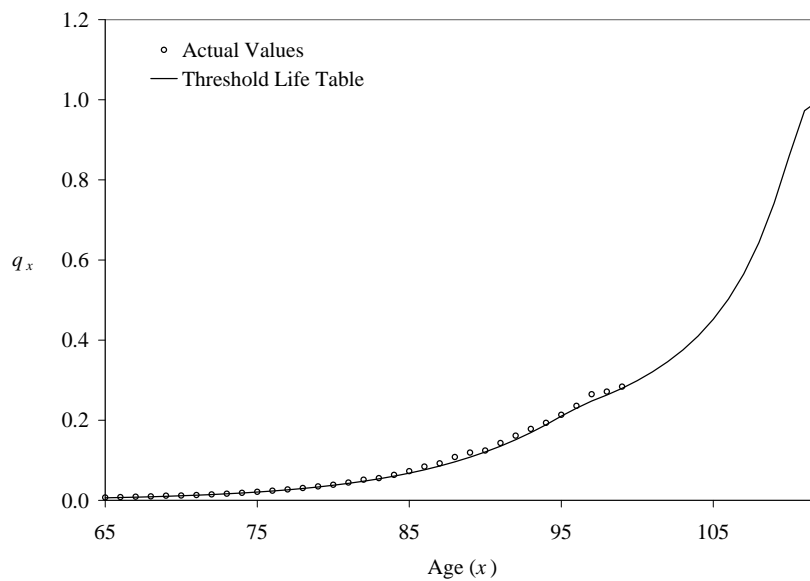


Figure 3. Estimated threshold life table for the population of New Zealand ($N=93$)

Figures 2 and 3 shows the maximum likelihood estimated threshold life tables for the populations of Australia and New Zealand, respectively. We observe a good fit over the entire life span. Even though the model specification does not guarantee smoothness at the threshold age N , we observe that the body and the tail join with each other fairly smoothly. This result arises because the estimation methods focus on the overall fitness, rather than the fitness at extreme ages only. Beyond the threshold age, the death probability increases progressively to 1 at the end point, which is determined statistically. We provide a further discussion on this issue in Section 4, which focuses on the prediction of the highest attained age.

4. THE LIMITING AGE

In the threshold life table, the exceedance $Z = \{X - N | X > N\}$ over the threshold age N follows a Generalized Pareto distribution with parameters γ and θ . The value of γ determines the precise distribution of Z . If $\gamma > 0$, then Z follows a Pareto distribution; if $\gamma = 0$, then Z follows an exponential distribution; if $\gamma < 0$, then Z follows a beta distribution, which has a finite right-hand-end support $-\theta/\gamma$. When the model is fitted to the mortality data for the two populations we consider, the values of γ are significantly less than zero. This means that the life tables for both populations have a theoretical end point that is given by $N - \theta/\gamma$. Note that $N - \theta/\gamma$ is strictly greater than N since $\theta > 0$ and $\gamma < 0$.

Now we consider a group of n individuals who survive to the threshold age N . We let X_i be the age at death for the i th individual, $Z_i = \{X_i - N | X_i > N\}$, and $M_n = \max\{Z_i, i = 1, 2, \dots, n\}$. The highest attained age for this group of n individuals can be expressed as $N + M_n$.

Assuming that Z_i and Z_j are independent if $i \neq j$ and that $Z_i, i = 1, \dots, n$, follows a Generalized Pareto distribution with $\gamma < 0$, the distribution function of M_n can be expressed as follows:

$$F_{M_n}(y) = \begin{cases} 0, & y < 0 \\ (F_Y(y))^n, & 0 \leq y < -\theta/\gamma. \\ 1, & y \geq -\theta/\gamma \end{cases}$$

It follows that when n tends to infinity, the distribution of M_n degenerates at $-\theta/\gamma$. Mathematically,

$$\lim_{n \rightarrow \infty} F_{M_n}(y) = \begin{cases} 0, & y < -\theta/\gamma \\ 1, & y \geq -\theta/\gamma. \end{cases}$$

In other words, if the number of survivors (n) at the threshold age is sufficiently large, the highest attained age of these n survivors converges (in probability) to $\omega = N - \frac{\theta}{\gamma}$, the theoretical end point of the threshold life table.

Note that the variability in the estimate of ω is inherited from those in the maximum likelihood estimates of θ and γ . Hence, the asymptotic variance of $\hat{\omega}$, the estimate of ω , can be computed by using the classical delta method (Klein, 1953, p.258), which can be summarized by the following equation:

$$\text{Var}(\hat{\omega}) = \begin{pmatrix} \frac{\partial \omega}{\partial \gamma} & \frac{\partial \omega}{\partial \theta} \end{pmatrix} (I(\gamma, \theta))^{-1} \begin{pmatrix} \frac{\partial \omega}{\partial \gamma} \\ \frac{\partial \omega}{\partial \theta} \end{pmatrix},$$

where $I(\gamma, \theta)$ is the information matrix for estimating γ and θ . Using the asymptotic normality property of maximum likelihood estimates, the approximate 95% confidence interval for ω can be written as

$$\left(\hat{\omega} - 1.96\sqrt{\text{Var}(\hat{\omega})}, \hat{\omega} + 1.96\sqrt{\text{Var}(\hat{\omega})} \right).$$

The mean and interval estimates of ω for both populations are displayed in Table 1.

In Table 2 we provide a list of validated supercentenarians (i.e., persons who have reached the age of 110 years or more) in Australia and New Zealand. This list is obtained from the Gerontology Research Group (www.grg.org). Our results are fairly consistent with the information given in Table 2. For Australia, the

central estimate of ω is very close to the actual highest attained age. For New Zealand, the actual highest attained age is included in the 95% interval estimate of ω .

5. CONCLUSION & FURTHER RESEARCH

Old-age mortality rates are important in many actuarial applications. This paper considers a method called the threshold life table, which is based on the asymptotic distribution of the exceedances over a threshold age. We apply this method to the populations of Australia and New Zealand. The resulting life tables are smooth and fit well to the raw mortality data.

We further extend the method to predict the highest attained age (ω) in Australia and New Zealand. On the basis of the threshold life table, the central estimates of ω for Australia and New Zealand are 112.20 and 109.43, respectively. Our estimates of ω are reasonably consistent with the validated supercentenarian in these countries.

In this paper, we rely on the asymptotic normality of maximum likelihood estimators to construct confidence intervals for ω . A problem of this method is that the resulting confidence intervals must be symmetric. A possible solution to this problem is Markov Chain Monte Carlo (MCMC). The use of MCMC to construct confidence intervals ω is a possible topic for further research.

Table 1. Predicted limiting ages

Country	Central estimate	95% Confidence interval
Australia	112.20	(108.09, 116.34)
New Zealand	109.43	(105.46, 113.40)

Table 2. Validated supercentenarians in Australia and New Zealand as of Nov. 22, 2008

Country	Name	Current status	Current age / age at death
Australia	E. Beatrice Riley	Alive	112
Australia	Myrtle Jones	Alive	111
Australia	Doreen Washington	Alive	110
Australia	Myra Nicholson	Died 2007	112
New Zealand	Florence Finch	Died 2007	113

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