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Calculation of the steady state probabilities for a water management problem with 3 connected dams

Paraskevi Thomas[1]

[1] *School of Mathematics and Statistics, University of South Australia, Australia*
Email: Paraskevi.Thomas@postgrads.unisa.edu.au

Abstract: We consider a system of three connected dams consisting of a supply, storage and capture dam, with capacities l, m and n, respectively. We assume a water management policy of a one unit supply from the supply dam, a pump-to-fill transfer policy between the dams and a random input of r units with probability p_r into the capture dam. We model the system as a discrete-time Markov chain with states $z = (z_3, z_2, z_1) \in S$ where z_3 is the discretised amount of water in the supply dam, z_2 of the storage dam and z_1 of the capture dam. Therefore we have $|S| = (l+1)(m+1)(n+1)$. The transition matrix $H \in \mathbb{R}^{(l+1)(m+1)(n+1)\times(l+1)(m+1)(n+1)}$ for this policy has the general block structure

$$H = \begin{pmatrix} A & \Sigma A & \Sigma^2 A & \cdots & \Sigma^{l-2} A & \Sigma^{l-1} A & A_l \\ A & \Sigma A & \Sigma^2 A & \cdots & \Sigma^{l-2} A & \Sigma^{l-1} A & A_l \\ 0 & A & \Sigma A & \cdots & \Sigma^{l-3} A & \Sigma^{l-2} A & A_{l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Sigma A & \Sigma^2 A & A_3 \\ 0 & 0 & 0 & \cdots & A & \Sigma A & A_2 \\ 0 & 0 & 0 & \cdots & 0 & A & A_1 \end{pmatrix}$$

where $A, A_i \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$ and also have a general block structure comprised of blocks of size $\mathbb{R}^{(n+1)\times(n+1)}$, and where Σ is a permutation matrix.

We denote the transient state probability vector as $\mathbf{x}_{(t)}^T$ and since the transition matrix is irreducible the system converges to a unique non-zero limiting probability distribution (or steady state probabilities) $\mathbf{x}^T$ such that

$$\mathbf{x}_{(t)}^T = \mathbf{x}_{(t-1)}^T H = \mathbf{x}_{(t-1)}^T = \mathbf{x}^T \quad \Rightarrow \quad \mathbf{x}^T = \mathbf{x}^T H.$$

The task becomes to solve $\mathbf{x}^T = \mathbf{x}^T H$ for $\mathbf{x}$, a left eigenvector problem for the eigenvalue $\lambda = 1$. Due to the large dimension of H this task is often difficult and time-consuming. Therefore we consider a reduction technique to reduce the problem to one of the order of the capacity of the capture dam.

Transforming the problem to $(I - K)\mathbf{x} = 0$, where $K = H^T$, the task becomes to solve for the null space of $(I - K)$. We apply Gaussian elimination to the block structure of $(I - K)$ and reduce the original eigenvector problem to solve the significantly smaller null space problem

$$(I - Z_m)\boldsymbol{\pi}_m = 0$$

where $Z_m \in \mathbb{R}^{(n+1)\times(n+1)}$. The remaining elements of the invariant state probability vector are then evaluated by the back-substitution process

$$\boldsymbol{\pi}_i = -Z_i \boldsymbol{\pi}_{i+1} \quad \text{and} \quad \mathbf{x}_j = -X_j \mathbf{x}_{j+1}$$

for $i = m-1, m-2, \ldots, 1, 0$ and $j = l-1, l-2, \ldots, 1, 0$.

We demonstrate the reduction technique with a small system and calculate the steady state probabilities in a numerical example.

We have observed a pattern for the reduction but have yet to derive a general formula for the reduced problem. Future work will also include modelling and reduction of larger, more complicated systems where the dimension of the transition matrix becomes even greater and the steady state probabilities more difficult to calculate.

Keywords: *3 connected dams, steady state probabilities, Markov chain, Gaussian elimination*

1. INTRODUCTION

Motivated by the problems faced by the Snowy Mountain Authority in Australia, Moran (1954, 1959) first published his work on the theory of dam storage. He considered a finite dam with random inputs and a regular demand and described a method for evaluating the probability distribution of the amount of water stored in the dam.

More recently Piantadosi (2004) considered the case of two connected dams, a capture and supply dam, with a regular demand of one unit from the supply dam and random inputs into the capture dam. Modelling the system as a Markov chain she derived a transition probability matrix with a general block structure to solve for the invariant state probability vector of the water stored in the dams. Using matrix reduction methods she proceeded to reduce the size of the problem to one of the order of the capacity of the supply dam.

In a similar approach we consider a system of three connected dams with random inputs into the first, a regular demand from the last and a pump-to-fill transfer policy and describe a general method for its reduction. The steady state probabilities are derived from a back-substitution process and may then be used to evaluate the risks associated with the system such as overflow and failure to satisfy the regular demand. An advantage of having the intermediate dam is the progressive cleaning of the water as it is pumped across the system, allowing impurities to settle in the dams.

We consider the first and intermediate dams as separate states, even though it may appear that we could combine their contents as a single state as we are using the pump-to-fill policy. However, there are cases where these two dams would not act like one larger dam, for example, when the input is large enough to overflow the first dam when the intermediate dam is not completely full. Also, this approach will allow us to build on the model in the future to consider extractions and inputs from the intermediate dam.

2. FORMULATION OF THE PROBLEM

We consider a system of three connected dams consisting of a supply, storage and capture dam with capacities l, m and n, respectively, where $l < m < n$. We assume a regular output of one unit from the supply dam at every stage, a pump-to-fill water transfer policy between the dams and a random input r into the capture dam. The general system is shown in Figure 1.

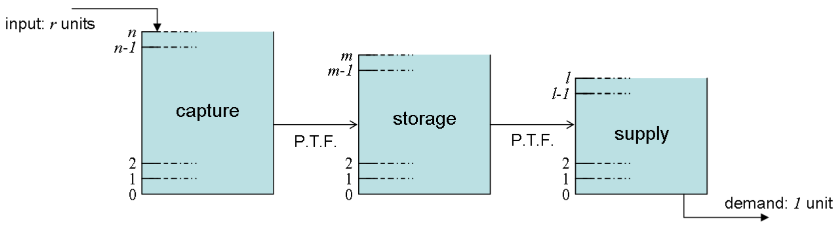

Figure 1. System of 3 connected dams

We model the system as a discrete-time Markov chain (for example, see Ross (1996)) where each state is defined as $z = (z_3, z_2, z_1) \in S$ (state space) where z_3 is the discretised amount of water in the supply dam, z_2 the amount in the storage dam and z_1 the amount in the capture dam. Thus we have $|S| = (l+1)(m+1)(n+1)$ as the total number of states. More precisely the state ordering is given by

$$\begin{aligned} S \;=\; & \{(0,0,0),(0,0,1),\ldots,(0,0,n),(0,1,0),(0,1,1),\ldots,(0,1,n),\ldots,(0,m,0),(0,m,1),\ldots,(0,m,n), \\ & (1,0,0),(1,0,1),\ldots,(1,0,n),(1,1,0),(1,1,1),\ldots,(1,1,n),\ldots,(1,m,0),(1,m,1),\ldots,(1,m,n), \\ & \ldots,(l,m,0),(l,m,1),\ldots,(l,m,n)\}\,. \qquad (1) \end{aligned}$$

The pump-to-fill water transfer policy between the dams aims to transfer as much water as possible to the downstream dams in an effort to maximise supply and minimise overflow from the capture dam. The policy operates with the following general rule: Suppose the amounts of water in the dams is given by state $z = (z_3, z_2, z_1)$ before transferring water. The transfer amounts become

- first transfer $u_3 = \min[z_2, l - z_3]$ from the storage to the supply dam, and
- then transfer $u_2 = \min[z_1, m - (z_2 - u_3)]$ from the capture to the storage dam.

Then the amounts of water in the dams after the transfers would be $z' = (z'_3, z'_2, z'_1)$ where

$$z'_3 = z_3 + u_3, \qquad z'_2 = z_2 - u_3 + u_2, \qquad \text{and} \qquad z'_1 = z_1 - u_2.$$

The probability that r units of water enters the capture dam at a given stage is denoted by $p_r > 0$. Also, we define the probability of at least s units entering the capture dam as $p_s^+ = \sum_{r=s}^{\infty} p_r$. Note that we must have $\sum_{r=0}^{\infty} p_r = 1$.

To determine the transition probabilities we assume the following order of operations during a stage for the water management policy of the system:

1. output of one unit from the supply dam if available
2. pump-to-fill transfers between the dams (as described above)
3. random input of r units into the capture dam with probability p_r.

With this water management policy and the state orderings given in equation (1) a general form of the stationary transition probability matrix $H \in \mathbb{R}^{(l+1)(m+1)(n+1)\times(l+1)(m+1)(n+1)}$ may be derived with the following general block structure

$$H = \begin{pmatrix} A & \Sigma A & \Sigma^2 A & \cdots & \Sigma^{l-2}A & \Sigma^{l-1}A & A_l \\ A & \Sigma A & \Sigma^2 A & \cdots & \Sigma^{l-2}A & \Sigma^{l-1}A & A_l \\ 0 & A & \Sigma A & \cdots & \Sigma^{l-3}A & \Sigma^{l-2}A & A_{l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Sigma A & \Sigma^2 A & A_3 \\ 0 & 0 & 0 & \cdots & A & \Sigma A & A_2 \\ 0 & 0 & 0 & \cdots & 0 & A & A_1 \end{pmatrix} \tag{2}$$

where $A, \Sigma, A_i \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$:

$$A = \begin{pmatrix} P_0 & P_1 & \cdots & P_{m-1} & P_m^+ \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ P_0 & P_1 & \cdots & P_m^+ \\ 0 & P_0 & \cdots & P_{m-1}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_i^+ \end{pmatrix}$$

for $i = 1, 2, \ldots, l$, and where $P_0, P_j, P_k^+, R, I \in \mathbb{R}^{(n+1)\times(n+1)}$:

$$P_0 = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_n^+ \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad P_j = R^j P_0 \quad \text{for } j = 1, 2, \ldots, n$$

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad P_k^+ = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_0 & p_1 & \cdots & p_n^+ \\ 0 & p_0 & \cdots & p_{n-1}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k^+ \end{pmatrix} \quad \text{for } k = 1, 2, \ldots, n-1.$$

We let $\mathbf{x}_{(t)}^T \in \mathbb{R}^{(l+1)(m+1)(n+1)}$ denote the transient state probability vector at stage t and is evaluated by

$$\mathbf{x}_{(t)}^T = \mathbf{x}_{(t-1)}^T H. \tag{3}$$

Therefore we have

$$\begin{aligned} \mathbf{x}_{(t)}^T &= \mathbf{x}_{(t-1)}^T H = \left(\mathbf{x}_{(t-2)}^T H\right) H = \mathbf{x}_{(t-2)}^T H^2 \\ &= \left(\mathbf{x}_{(t-3)}^T H\right) H^2 = \mathbf{x}_{(t-3)}^T H^3 \\ &= \cdots = \mathbf{x}_{(0)}^T H^t \end{aligned}$$

where $\mathbf{x}_{(0)}^T$ denotes the probability of the initial state of the system.

3. SOLUTION PROCEDURE

Due to the general structure of the transition matrix describing an irreducible Markov chain it follows that $\lim_{t\to\infty} H^t$ exists and that the transient state probability vector $\mathbf{x}_{(t)}^T$ converges to a unique non-zero limiting probability distribution $\mathbf{x}^T$ (also known as the invariant state probability vector, or steady state probabilities) (Ross, 1996). Hence we have the system converging when

$$\mathbf{x}_{(t)}^T = \mathbf{x}_{(t-1)}^T H = \mathbf{x}_{(t-1)}^T = \mathbf{x}^T.$$

Thus the problem becomes to solve

$$\mathbf{x}^T = \mathbf{x}^T H \tag{4}$$

for $\mathbf{x}^T$, a left eigenvector problem with eigenvalue $\lambda = 1$, under the normalisation $\mathbf{x}^T \mathbf{1} = 1$.

We may expand the definition of the invariant state probability vector to

$$\mathbf{x}^T = \left[\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_l^T\right] \tag{5}$$

where $\mathbf{x}_i^T$ is the invariant state probability vector for the states (i, z_2, z_1), for $z_2 = 0, 1, \ldots, m$ and $z_1 = 0, 1, \ldots, n$.

When H is large then the problem in equation (4) becomes difficult and time-consuming to solve. We apply a reduction technique to reduce the size of the problem from $\mathbb{R}^{(l+1)(m+1)(n+1)\times(l+1)(m+1)(n+1)}$ to $\mathbb{R}^{(n+1)\times(n+1)}$.

3.1. Reduction procedure

We begin by transforming the problem to obtain

$$\mathbf{x}^T = \mathbf{x}^T H \quad \Leftrightarrow \quad \mathbf{x} = H^T \mathbf{x} \quad \Leftrightarrow \quad (I - K)\mathbf{x} = 0$$

where $K = H^T$ and we let $C_i = P_i^T$ and $C_i^+ = (P_i^+)^T$. The task now becomes to solve for the null space of $(I - K)$ to find the invariant state probability vector $\mathbf{x}$.

The matrix $(I - K)$ may be divided into $J_{i,j}$ blocks where $J_{i,j} \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$ and $i, j = 0, 1, \ldots, l$. Therefore the problem $(I - K)\mathbf{x} = 0$ may be written as

$$\begin{pmatrix} J_{0,0} & J_{0,1} & \cdots & J_{0,l} \\ J_{1,0} & J_{1,1} & \cdots & J_{1,l} \\ \vdots & \vdots & \ddots & \vdots \\ J_{l,0} & J_{l,1} & \cdots & J_{l,l} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

We then systematically apply Gaussian elimination (for example, see Hill (1996)) to the J-block structure in equation (6), using pre-multiplication of matrices, with the following general procedure to reduce $(I - K)$ to block-row-echelon form:

- make the diagonal elements $J_{i,i} = I$ for $i = 0, 1, \ldots, l - 1$, and
- make the elements under the diagonal $J_{j,k} = 0$ for $j > k$ and $k \neq l$.

Note that the elimination procedure requires inverting some key matrices of the form $(I - B)$ where B is a sub-stochastic matrix, thus the inverse of $(I - B)$ is well-defined such that $(I - B)^{-1} = \sum_{i=0}^{\infty} B^i$ exists.

Therefore $(I - K)$ may be reduced to

$$I - K \sim \begin{pmatrix} I & X_0 & 0 & 0 & \cdots & 0 \\ 0 & I & X_1 & 0 & \cdots & 0 \\ 0 & 0 & I & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I & X_{l-1} \\ 0 & 0 & 0 & \cdots & 0 & I - X_l \end{pmatrix} \tag{7}$$

where $X_0, X_i \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$ for $i = 1, 2, \ldots, l - 1$ with only the first block-column consisting of non-zero elements $\in \mathbb{R}^{(n+1)\times(n+1)}$ of the form

$$X_{0,j} = -C_j - \alpha_j C_0 \tag{8}$$

for $j = 0, 1, \ldots, m-1$ where $\alpha_j = \frac{p_j}{1-p_0}$ and

$$X_{0,m} = -C_m - \alpha_m C_0 - \sum_{k=1}^{n-m-1} \left[C_{m+k,k} + \alpha_{m+k} C_{0,k}\right] - C_{n,n-m} - \alpha_n^+ C_{0,n-m} \tag{9}$$

where $\alpha_j^+ = \frac{p_j^+}{1-p_0}$ and

$$X_{i,j} = -C_j - \beta_{i,j}\left(p_i C_0 + p_i^+ C_1\right) \tag{10}$$

where $\beta_{i,j} = \frac{p_j}{p_i^+(1-p_1)-p_0 p_i}$, and

$$\begin{aligned} X_{i,m} &= -C_m - \beta_{i,m}\left(p_i C_0 + p_i^+ C_1\right) - \sum_{k=1}^{n-m-1} \left[C_{m+k,k} + \beta_{i,m+k}\left(p_i C_{0,k} + p_i^+ C_{1,k}\right)\right] \\ &\quad -C_{m,n-m} - \beta_{i,n}^+\left(p_i C_{0,n-m} + p_i^+ C_{1,n-m}\right) \end{aligned} \tag{11}$$

where $\beta_{i,j}^+ = \frac{p_j^+}{p_i^+(1-p_1)-p_0 p_i}$ and the matrix $C_{i,j} \in \mathbb{R}^{(n+1)\times(n+1)}$ consists of the column vector $[p_0, p_1, \ldots, p_{n-j-1}, p_{n-j}^+]^T$ after i columns and j rows of zeros.

The problem is now reduced to solve

$$(I - X_l)\mathbf{x}_l = 0 \tag{12}$$

for $\mathbf{x}_l$ which is still quite large and difficult to solve. However, the reduced $J_{l,l} \sim (I - X_l)$ can be treated as an intermediate step in the reduction and may be reduced even further as described in the next section.

3.2. Further reduction

The reduced matrix $J_{l,l} \sim I - X_l \in \mathbb{R}^{(m+1)(n+1)\times(m+1)(n+1)}$ is comprised of blocks of the order $\mathbb{R}^{(n+1)\times(n+1)}$ and therefore we may apply Gaussian elimination to the block structure to reduce it even further to the form

$$I - X_l \sim \begin{pmatrix} I & Z_0 & 0 & \cdots & 0 \\ 0 & I & Z_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & Z_{m-1} \\ 0 & 0 & \cdots & 0 & I - Z_m \end{pmatrix} \tag{13}$$

where $Z_i \in \mathbb{R}^{(n+1)\times(n+1)}$ and we propose they are of the form $Z_i = -\frac{N_i}{D_i} C_0$ for $i = 0, 1, \ldots, m-1$ and $N_j = D_{j-1}$ and $D_j = (1-p_1)N_j - p_0 G_j$ for $j = 1, 2, \ldots, m-1$ where $N_i, D_i, G_i \in \mathbb{R}$. Due to the general structure of special matrices of the original transition probability matrix it follows that the reduced problem will also have a general formula but has not been derived as yet though there appears to be a pattern emerging in the examples.

If we define $\mathbf{x}_l^T$ as

$$\mathbf{x}_l^T = \left[\boldsymbol{\pi}_0^T, \boldsymbol{\pi}_1^T, \ldots, \boldsymbol{\pi}_m^T\right] \tag{14}$$

where $\boldsymbol{\pi}_i \in \mathbb{R}^{(n+1)}$ we have the previous reduction in equation (12) reduced to the problem

$$(I - Z_m)\boldsymbol{\pi}_m = 0. \tag{15}$$

Since $(I - K)$ is non-singular (as the solution to equation (4) for $\mathbf{x}$ is not trivial) it follows that $(I - Z_m)$ is non-singular and a unique solution for $\boldsymbol{\pi}_m$ exists.

3.3. Back-substitution process

Once $\boldsymbol{\pi}_m$ has been determined from solving equation (15) the remaining $\boldsymbol{\pi}_i$ probabilities may be found by the back-substitution process from the matrices in equation (13) with

$$\boldsymbol{\pi}_i = -Z_i \boldsymbol{\pi}_{i+1} \tag{16}$$

for $i = m-1, m-2, \ldots, 1, 0$ giving $\mathbf{x}_l^T = \left[\boldsymbol{\pi}_0^T, \boldsymbol{\pi}_1^T, \ldots, \boldsymbol{\pi}_m^T\right]$. Then the remaining $\mathbf{x}_j$ probabilities may also be evaluated by the back-substitution process defined by the matrix in equation (7):

$$\mathbf{x}_j = -X_j \mathbf{x}_{j+1} \tag{17}$$

for $j = l-1, l-2, \ldots, 1, 0$ to determine the invariant state probability vector $\mathbf{x}^T = \left[\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_l^T\right]$.

4. EXAMPLE $l = 2, m = 3, n = 5$

We consider a small example to illustrate our reduction procedure where $l = 2$, $m = 3$ and $n = 5$. Then the transition probability matrix $H \in \mathbb{R}^{72\times 72}$ is given by

$$H = \begin{pmatrix} A & \Sigma A & A_2 \\ A & \Sigma A & A_2 \\ 0 & A & A_1 \end{pmatrix}$$

where

$$A = \begin{pmatrix} P_0 & P_1 & P_2 & P_3^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P_0 & P_1 & P_2 & P_3^+ \\ 0 & P_0 & P_1 & P_2^+ \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ P_0 & P_1 & P_2 & P_3^+ \\ 0 & P_0 & P_1 & P_2^+ \\ 0 & 0 & P_0 & P_1^+ \end{pmatrix}$$

and some of the P_i, P_i^+ matrices are

$$P_0 = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5^+ \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p_0 & p_1 & p_2 & p_3 & p_4 & p_5^+ \\ 0 & p_0 & p_1 & p_2 & p_3 & p_4^+ \\ 0 & 0 & p_0 & p_1 & p_2 & p_3^+ \\ 0 & 0 & 0 & p_0 & p_1 & p_2^+ \end{pmatrix}.$$

Then, by Gaussian elimination $\mathbf{x}^T = \mathbf{x}^T H$ is reduced to

$$I - K \sim \begin{pmatrix} I & X_0 & 0 \\ 0 & I & X_1 \\ 0 & 0 & I - X_2 \end{pmatrix} \quad \text{and} \quad I - X_2 \sim \begin{pmatrix} I & Z_0 & 0 & 0 \\ 0 & I & Z_1 & 0 \\ 0 & 0 & I & Z_2 \\ 0 & 0 & 0 & I - Z_3 \end{pmatrix}$$

where

$$X_0 = \begin{pmatrix} X_{0,1} & 0 & 0 & 0 \\ X_{0,2} & 0 & 0 & 0 \\ X_{0,3} & 0 & 0 & 0 \\ X_{0,4} & 0 & 0 & 0 \end{pmatrix}, \; X_1 = \begin{pmatrix} X_{1,1} & 0 & 0 & 0 \\ X_{1,2} & 0 & 0 & 0 \\ X_{1,3} & 0 & 0 & 0 \\ X_{1,4} & 0 & 0 & 0 \end{pmatrix}$$

as defined in equations (8), (9), (10), (11).

The final Z-matrices are

$$\begin{aligned} Z_0 &= -\frac{N_0}{D_0} C_0 \\ Z_1 &= -\frac{D_0}{(1-p_1)D_0 - p_0 G_1} C_0 \\ Z_2 &= -\frac{(1-p_1)D_0 - p_0 G_1}{(1-p_1)^2 D_0 - p_0(1-p_1)G_1 - p_0 G_2} C_0 \end{aligned}$$

where $N_0 = p_2^+$, $D_0 = p_2^+ - p_0 p_2$, $G_1 = p_0 p_3 + p_1 p_2$ and $G_2 = p_2 D_0 + p_0(p_0 p_4 + p_1 p_3 + p_2 p_2)$, and

$$\begin{aligned} I - Z_3 \;=\; & I - C_1^+ - \frac{1}{D_2}\big\{ \big[p_2 N_2 + p_0[p_3 D_0 + p_0(p_0 p_5^+ + p_1 p_4 + p_2 p_3 + p_3 p_2)]\big] C_0 \\ & + \big[p_3 N_2 + p_0[p_4 D_0 + p_0(p_1 p_5^+ + p_2 p_4 + p_3 p_3 + p_4 p_2)]\big] C_{0,1} \\ & + \big[p_4 N_2 + p_0[p_5^+ D_0 + p_0(p_2 p_5^+ + p_3 p_4 + p_4 p_3 + p_5^+ p_2)]\big] C_{0,2} \\ & + \big[p_5^+ N_2 + p_0 p_0 (p_3 p_5^+ + p_4 p_4^+ + p_5^+ p_3^+)\big] C_{0,3} \big\} \end{aligned}$$

where $D_2 = (1-p_1)^2 D_0 - p_0(1-p_1)G_1 - p_0 G_2$ and $N_2 = (1-p_1)D_0 - p_0 G_1$.

Then the problem becomes to solve

$$(I - Z_3)\boldsymbol{\pi}_3 = 0 \tag{18}$$

of the order $\mathbb{R}^{6\times 6}$ and to calculate the remaining invariant state probabilities by the back-substitution process $\boldsymbol{\pi}_i = -Z_i \boldsymbol{\pi}_{i+1}$ and $\mathbf{x}_j = -X_j \mathbf{x}_{j+1}$ for $i = 2, 1, 0$ and $j = 1, 0$.

Consider a numerical example with $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{8}$, $p_3 = \frac{1}{16}$, $p_4 = \frac{1}{32}$ and $p_5^+ = \frac{1}{32}$ where the average input is one unit, the same as the regular demand, and is hence a balanced system. Then to find the steady state probabilities we must solve the null space of

$$I - Z_3 \sim \begin{pmatrix} 3/4 & -1/2 & 0 & 0 & 0 & 0 \\ -1/4 & 3/4 & -1/2 & 0 & 0 & 0 \\ -49/240 & -1/8 & 3/4 & -1/2 & 0 & 0 \\ -71/480 & -1/16 & -1/8 & 3/4 & -1/2 & 0 \\ -71/960 & -1/32 & -1/16 & -1/8 & 3/4 & -1/2 \\ -71/960 & -1/32 & -1/16 & -1/8 & -1/4 & 1/2 \end{pmatrix}$$

giving $\boldsymbol{\pi}_3 = \left[\frac{314}{1275}, \frac{157}{425}, \frac{690}{1601}, \frac{625}{1378}, \frac{625}{1378}, \frac{625}{1378}\right]^T$ with all 72 of the normalised steady state probabilities of the system derived from the back-substitution process displayed in Figure 2. We can see from the plot that the system is in states $(2, 3, i)$ for $i = 0, 1, \ldots, 5$ more often than the other states which is expected under a 'pump to fill the downstream dams' transfer policy. Also there are high probabilities of being in states $(j, k, 0)$ for $j = 0, 1, \ldots, l$ and $k = 0, 1, \ldots, m$ which indicates that the capture dam is often empty and is able to capture as much water as it possibly can and minimise overflow – again a feature of the pump-to-fill policy.

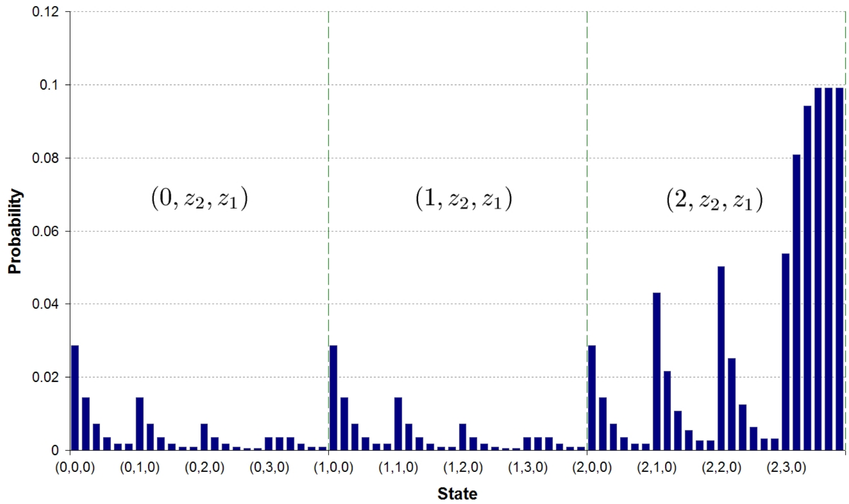

Figure 2. Steady state probabilities of a system with capacities $l = 2$, $m = 3$ and $n = 5$ and input probabilities $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{8}$, $p_3 = \frac{1}{16}$, $p_4 = \frac{1}{32}$ and $p_5^+ = \frac{1}{32}$ with the state ordering given by equation (1).

5. CONCLUSION

For a system of three connected dams modelled as a Markov chain the problem to solve for the invariant state probability vector $\mathbf{x}^T = \mathbf{x}^T H$ is often of large dimension and difficult. Recognising a general pattern in the transition matrix H the problem may be reduced significantly using Gaussian elimination to a similar problem but of the order of the capacity of the capture dam and the remaining steady state probabilities calculated by a back-substitution process.

A general process for the reduction of the problem has been described and even though a general formula for the complete reduction has yet to be derived there are patterns emerging in the examples. Future work includes deriving the general formula and the modelling and reduction of larger systems with variations to the input and release rules.

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