19th International Congress on Modelling and Simulation, Perth, Australia, 12–16 December 2011
http://mssanz.org.au/modsim2011

The Empirical Properties of Some Popular Estimators of Long Memory Processes

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Abstract: The empirical properties of 12 different estimators of the Hurst parameter, H, or fractional integration parameter, d, derived via simulation, are presented. For time series with fewer than 4,000 observations only the Whittle and Haslett-Raftery estimators produce acceptable statistical properties.

Keywords: Strong dependence, global dependence, long range dependence, long memory, Hurst parameter estimators

1 INTRODUCTION

The subject of long-memory time series has received extensive attention in both the statistical and econometric literature particularly financial econometrics. See Beran (1994), Robinson (2003) and Palma (2007).

Of critical importance in modeling long memory time series is estimating the strength of the long-range dependence. Two measures are commonly used. The parameter H, known as the Hurst or self-similarity parameter, was introduced by Mandelbrot and van Ness (1968). The fractional integration parameter, d, arises from the generalization of the Box-Jenkins ARIMA(p,d,q) models from integer to non-integer values of the integration parameter d. This generalization was accomplished independently by Granger and Joyeux (1980) and by Hosking (1981). The parameters H and d are related by the formula $H = d + 1/2$.

A number of estimators of H and d have been developed and theoretical results produced. Palma (2007, Chapters 4 and 5) gives some details on their asymptotic properties and further references to the literature.

While the asymptotic properties have been established for a number of these estimators, their properties may be quite different in finite series of lengths typical in empirical economic and financial research.

Taqqu et al. (1995) studied nine estimators for a single series length of 10,000 data points, five values of both H and d, and 50 replications. Jeong et al. (2007) compared six estimators on simulated series with 32,768 (2^{15}) observations, five values of H and 100 replications.

In the results reported here we extend these studies to include a larger number of parameters, higher number of replications and 12 estimators as detailed in Section (2) below.

The remainder of the paper is organized as follows. Section (2) gives details of the method. Section (3) presents the results. Section (4) gives our conclusions.

2 METHOD

Ten estimators are implemented in the package `fSeries` of Wuertz (2005) for the statistical software `R` (R Development Core Team, 2005). They are the absolute value, aggregated variance, boxed periodogram, differenced variance, Higuchi, Peng, periodogram, rescaled range, wavelet, and the Whittle. Further, the GPH (Geweke and Porter-Hudak, 1983) and Haslett and Raftery (1989) estimators for d is implemented in the contributed package `fracdiff` of Fraley et al. (2006).

Taqqu et al. (1995) simulated both fractional Gaussian noises (FGNs) and the corresponding discrete time fractionally integrated (FI(d)) series and found that each estimator performed similarly whether estimating H in simulated FGNs or d in simulated FI(d)s. For example, if an estimator was biased when estimating H it was also biased in a very similar manner when estimating d. Thus, for the 10 estimators implemented in `fSeries` we only investigated each estimator's performance in estimating H for simulated FGNs generated using the function `fgnSim` in `fSeries`. For the GPH and Haslett-Raftery estimators of d we generated FI(d) series with the function `farimaSim` in `fSeries`. We ran 1000 replications of simulated FGNs with 100 different lengths and eight different H values. The lengths were between 100 and 10,000 data points in steps of 100. The H values were between 0.55 and 0.90, or the equivalent d values, in steps of 0.05. For each series H was estimated by the ten estimators implemented in `fSeries` and d by the two estimators implemented in `fracdiff`. For each H value and series length we estimated the median, 75% and 95% confidence intervals empirically from the simulated data. The H or d estimates were sorted into ascending order and the median obtained by averaging the 500th and 501st values. Similar calculations were done for the upper and lower values of the 75% and 95% confidence intervals.

In the presentation of the results we converted the GPH and Haslett-Raftery d estimates to H equivalents to facilitate comparisons among the estimators.

3 RESULTS

For reasons of space we only present a few representative results. Figures (1) through (6) are presented with the vertical axis with a range of 1.2 H units to facilitate comparisons among the estimators' standard deviation of their estimates. It should be noted that stationary long memory occurs in the range $0.5 < H < 1.0$.

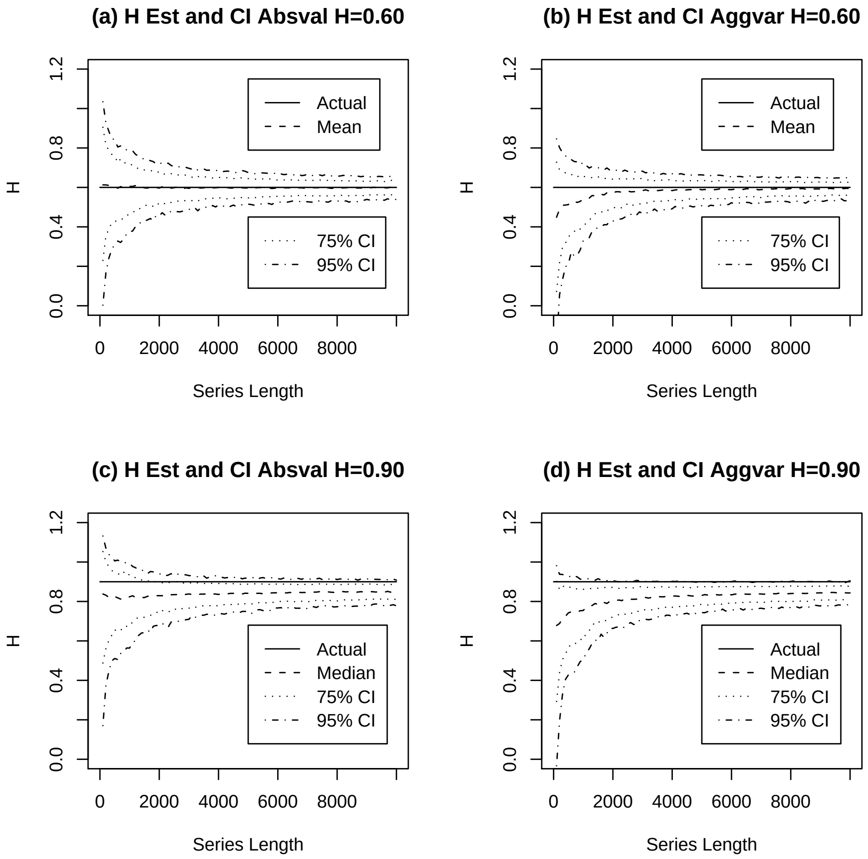

Figure 1. Empirical confidence intervals for the H estimates with $H = 0.60$ and $H = 0.90$; (a) and (c) absolute value method, (b) and (d) aggregated variance estimator.

The results for the absolute value of the variance method are presented in Figures (1) (a) and (c). It was unbiased at all series lengths when H was low (0.55 or 0.60) but became progressively biased and underestimated H as H increased.

The results for the aggregated variance method are presented in Figures (1) (b) and (d). It exhibited bias and underestimated H in short series when H was low. As H increased the estimator became increasingly biased at all series lengths examined. With $H = 0.90$ the true value of H lay above the upper 95% empirical confidence interval for all but the shortest series lengths.

The results for the boxed periodogram method are presented in Figures (2) (a) and (c). This estimator was developed specifically to deal with perceived problems with the periodogram estimator. It was biased towards underestimating H for almost all values of H and series lengths examined.

The results for the differenced variance method are presented in Figures (2) (b) and (d). This estimator had one of largest confidence intervals of the estimators when the series were short but this slowly decreased as sample size increased. The difference variance estimator exhibited bias towards over estimating H for any series with less than 7,000 observations. The bias was very serious in the short series. For series longer than about 9,000 observations the estimator exhibited a small amount of bias towards underestimating H.

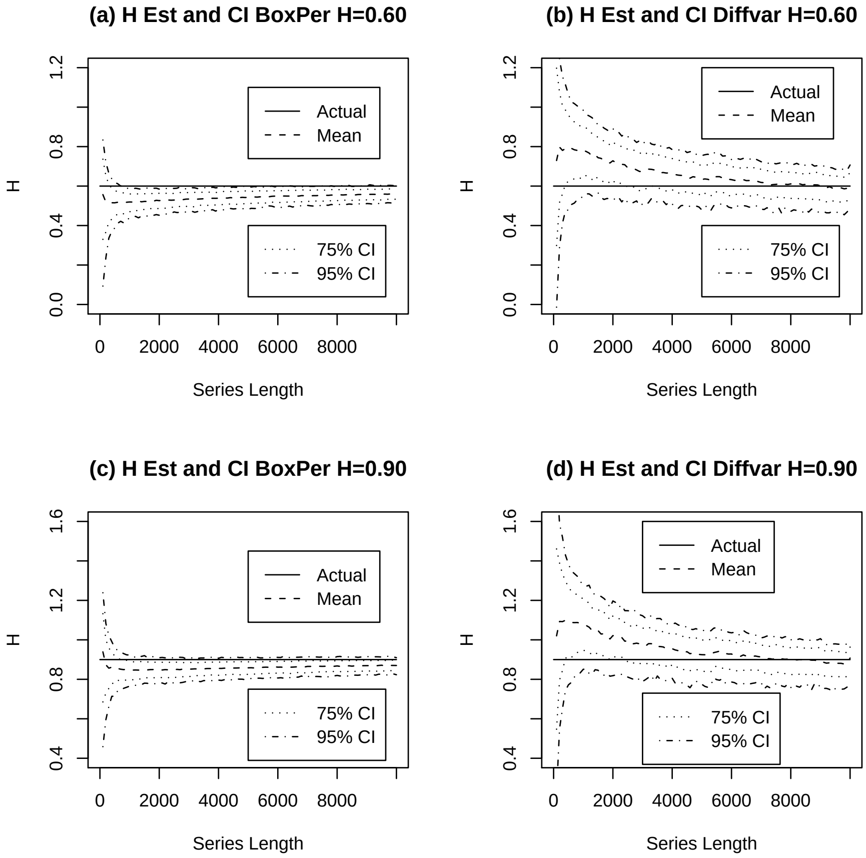

Figure 2. Empirical confidence intervals for the H estimates with $H = 0.60$ and $H = 0.90$; (a) and (c) boxed periodogram method, (b) and (d) differenced variance estimator.

The results for the Higuchi (1988) estimator are presented in Figures (3) (a) and (c). It was biased towards underestimating H but the magnitude of the bias appeared relatively independent of H. The width of the confidence interval of the estimate increased with increasing H.

The results for the Peng et al. (1994) estimator are presented in Figures (3) (b) and (d). It was biased toward under estimating H in the series lengths we investigated. This bias appeared to be independent of H but greater in short series.

The results for the R/S estimator are presented in Figures (4) (b) and (d). The R/S estimator exhibited three problems; it was biased upwards when H was low, it was biased downwards when H was high, and the confidence interval of the estimate did not decrease with increasing series length once the series reached about 1000 observations.

The results for the Whittle estimator are presented in Figures (5) (a) and (c). Compared to the other nine estimators implemented in `fSeries` the Whittle estimator was remarkable for its narrow confidence interval.

The results for the wavelet estimator are presented in Figures (5) (b) and (d). It was unbiased for all H values at series lengths over 4,100 data points. The bias present in series shorter than 4,100 data points was very small. The availability of a new *octave* can be seen in Figures (5) (b) and (d) with each doubling of the series length. New *octaves* resulted in a series of steps in the reduction of the confidence interval of the estimate with increasing series length. The estimator had constant variance when the number of

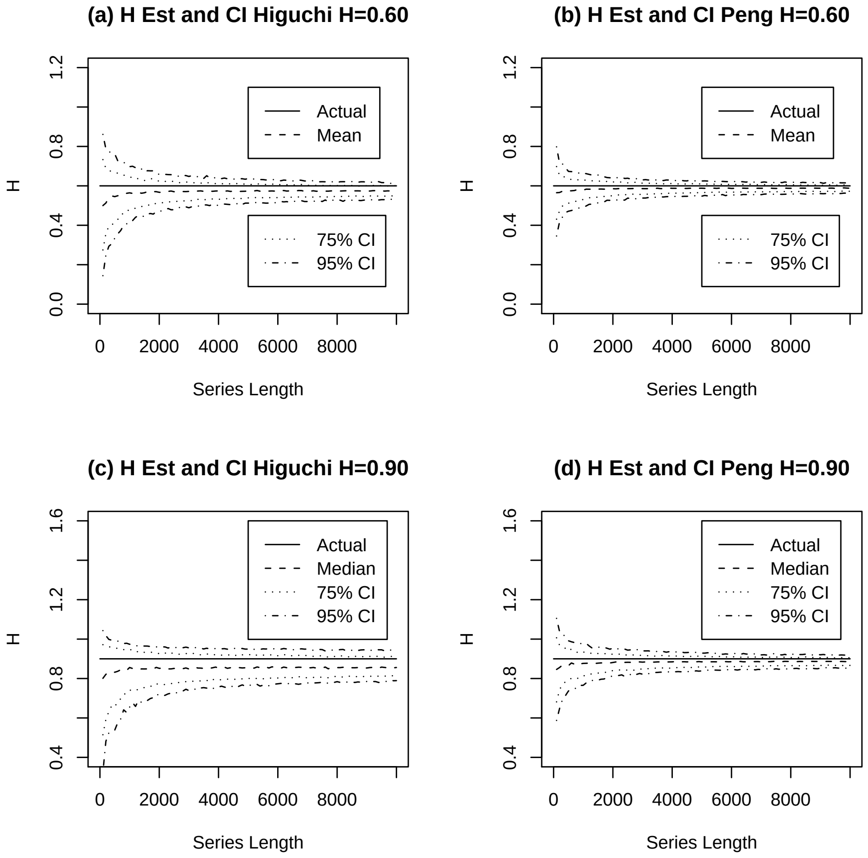

Figure 3. Empirical confidence intervals for the H estimates with $H = 0.60$ and $H = 0.90$; (a) and (c) Higuchi estimator, (b) and (d) Peng estimator.

octaves was constant.

The results for the GPH estimator are presented in Figures (6)(a) and (c). It exhibited a very small amount of bias towards overestimating d at all series lengths examined. It had a very wide confidence interval which narrowed slowly as the series length increased.

The results for the Haslett-Raftery estimator are presented in Figures (6)(b) and (d). The Haslett-Raftery did not report estimates of d less than zero ($H < 0.5$). Hence for low d and short series the distribution was truncated on the low side at $d = 0$ or $H = 0.5$ as in Figure (6) (a). The Haslett-Raftery estimator was an excellent estimator with only small amounts of bias in the short series and had a narrow confidence interval.

4 CONCLUSIONS

Of the twelve estimators examined here the Whittle and Haslett-Raftery estimators performed the best on simulated series. If we require the estimators to be unbiased and with a 95 percent confidence interval width of less than 0.1 H or d units, then for series with less than 4,000 data points they were the only two estimators worth considering. It should be noted that these estimators did not meet these criteria until the series lengths exceeded 700 and 1000 data points respectively. For series with 4,000 or more data points, the Peng estimator gave acceptable performance. For series with more than 7,000 data points the periodogram estimator was a worthwhile choice. For series with more than 8,200 data points the wavelet

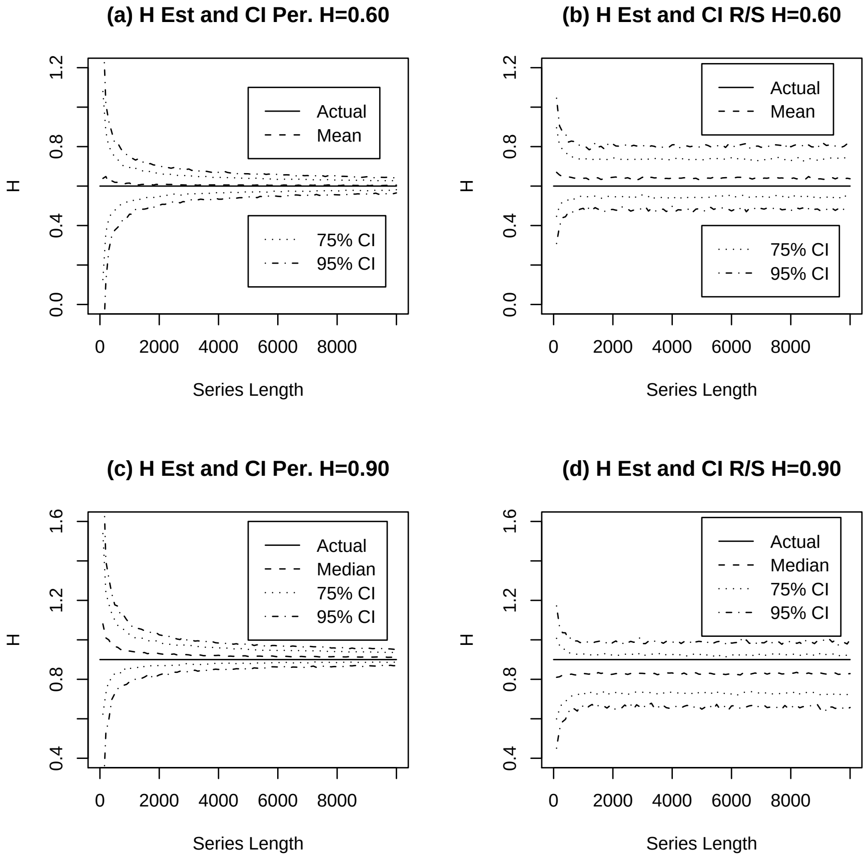

Figure 4. Empirical confidence intervals for the H estimates with $H = 0.60$ and $H = 0.90$; (a) and (c) periodogram estimator, (b) and (d) R/S estimator.

estimator became a viable estimator. The remaining seven estimators did not give acceptable performance at any series lengths examined and are not recommended.

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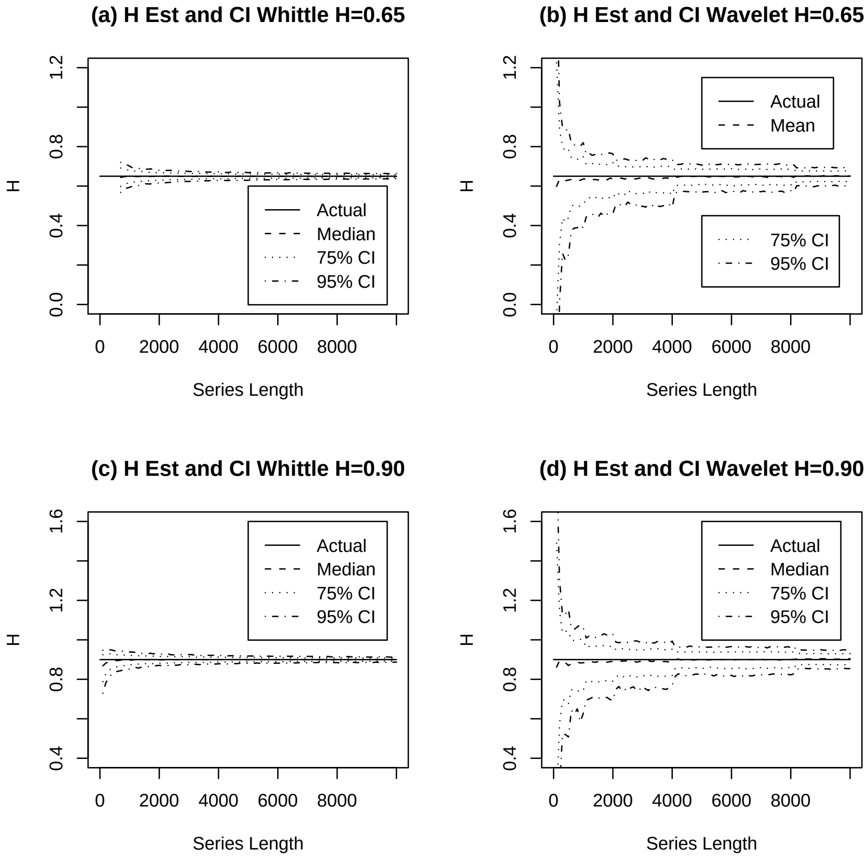

Figure 5. Empirical confidence intervals for the H estimates with $H = 0.65$ and $H = 0.90$; (a) and (c) Whittle estimator, (b) and (d) wavelet estimator.

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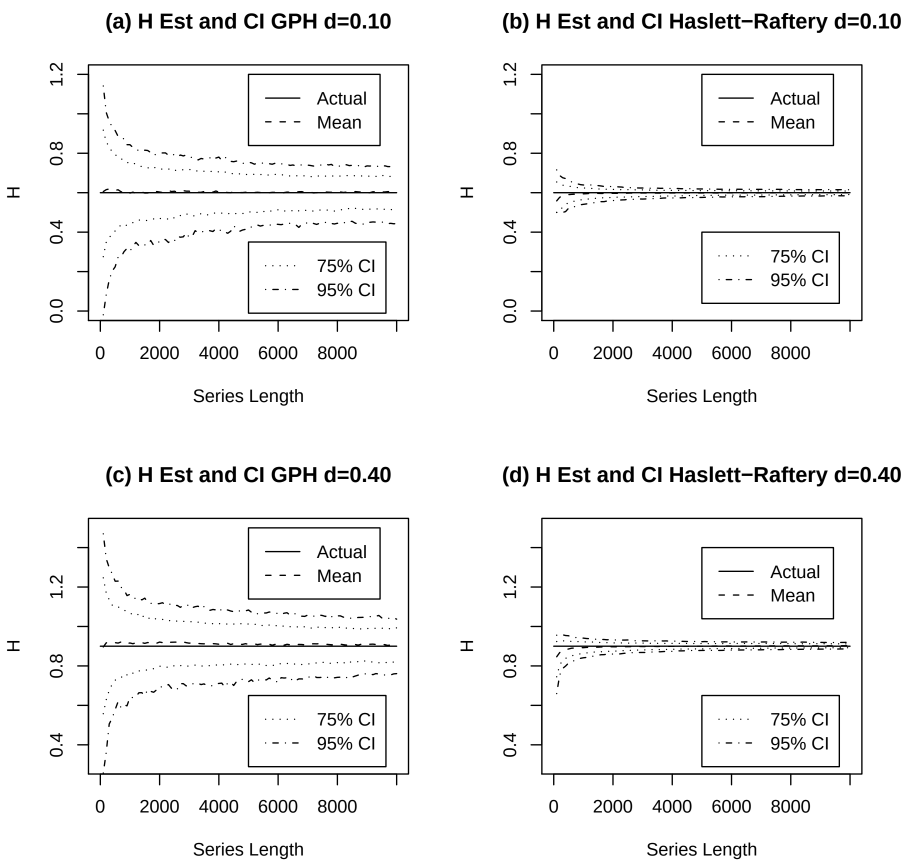

Figure 6. Empirical confidence intervals for the H estimates with $d = 0.10$ ($H = 0.60$) and $d = 0.40$ ($H = 0.90$); (a) and (c) GPH estimator, (b) and (d) Haslett-Raftery estimator.