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A Symmetry Analysis of Non-Autonomous von Bertalanffy Equations

M. P. Edwards [a], **R. S. Anderssen** [b]

[a] *School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW, 2522*
[b] *CSIRO Computational Informatics, GPO 664, Canberra, Australian Capital Territory, 2601*
Email: maureen@uow.edu.au, Bob.Anderssen@csiro.au

Abstract: In industrial applications involving microbial growth including food contamination, it is often the number of surviving microbes that must be modelled not the total population. For the total population N situation, the von Bertalanffy equation (von Bertalanffy (1957))

$$\frac{dN}{dt} = \alpha N^{\beta} - aN^{b}, \quad N(0) = N_0, \quad N = N(t), \tag{1}$$

has been a popular choice, because it models the situation as a competition between growth and decay. In addition, for various choices for the parameters α, β, a and b, analytic solutions are known which avoids the necessity to determine N using numerical methods. However, the morphology of its solutions is limited to being strictly monotonic increasing or decreasing, or asymptoting monotone increasing or decreasing functions. Consequently, this class of functions does not include functions which first increase, attain a maximum and then decay, which corresponds to the dynamics of the number N of surviving microbes in a closed environment. In order to model such situations as a single ordinary differential equation in N, the dynamics must be defined in terms of an appropriate non-autonomous ordinary differential equation with the non-autonomous terms taking account of the effect of the environment as well as the current size of the surviving population.

In a recent paper by Edwards et al. (2013), non-autonomous versions of the autonomous von Bertalanffy equation have been proposed to model microbial growth in a closed environment. The role played by the non-autonomous terms in this equation is discussed. The non-autonomous equation is extended here to include an additional non-autonomous term and a symmetry analysis has been used to study the structure of the solutions that such equations can generate.

Here it is shown how the non-autonomous von Bertalanffy equation can be reduced to an integrable equation for specific forms of the arbitrary functions.

Keywords: Autonomous, non-autonomous, ordinary differential equations, von Bertalanffy, symmetry analysis

1 INTRODUCTION

In modelling real world processes, the choice of the mathematical model represents a balancing of writing down a representative model and having a model for which a mathematical analysis gives insight. de Hoog (2009) suggests that, in part, a strategy for achieving this is to keep the model as simple as possible.

Such a situation arises in the modelling of microbial growth.

Here, the choice is between autonomous and non-autonomous models. In one dimension, as explained in Edwards et al. (2013), a clear difference between what can be modelled with a single autonomous equation is not as general as that which can be modelled using the corresponding non-autonomous version

$$\frac{dN}{dt} = \alpha(t)N^{\beta} - a(t)N^{b}, \quad N = N(t), \tag{2}$$

of the von Bertalanffy (1957) equation. The non-autonomous equation (2) is extended to include an additional term $\psi(t)$, so that

$$\frac{dN}{dt} = \alpha(t)N^{\beta} - a(t)N^{b} + \psi(t), \quad N = N(t), \tag{3}$$

where $N(t)$ models the dynamics of the population size (number) as a function of the time t with $\alpha(t)$, β, $a(t)$ and b all non-negative. The equivalent autonomous counterpart of this equation is given by

$$\frac{dN}{dt} = \alpha(z)N^{\beta} - a(z)N^{b} + \psi(z), \quad N = N(z), \tag{4a}$$

$$\frac{dz}{dt} = 1. \tag{4b}$$

This illustrates that any coupled system of n non-autonomous ODEs can be rewritten as a coupled system of $(n+1)$ autonomous ODEs. The dimension of the Lie symmetry algebra admitted by the first-order equation (3) is infinite. However, the dimension of the Lie symmetry algebra admitted by a system of n first-order equations is also infinite, and there is no obvious advantage in considering the autonomous system (4).

From a real modelling perspective, it is the non-autonomous terms which drive the dynamics and allows a simple model to be formulated whereas the corresponding autonomous version is a coupled system of ODEs.

2 THE ESSENCE OF SYMMETRY ANALYSIS

It was known historically that non-linear differential equations could be solved analytically by introducing a judiciously chosen transformation that mapped the given equation of interest into an equivalent (invariant) equation (i.e. an equation with the same solutions but in a different algebraic form), for which analytic solutions were known or could be determined. The prototypical example is the Bernoulli equation

$$\frac{dy}{dx} = f_n(x)y^n + f_1(x)y \tag{5}$$

for which the transformation $v = y^{1-n}$ takes it to the linear form

$$\frac{dv}{dx} - (1-n)f_1(x)v = (1-n)f_n(x).$$

Solving this equation and eliminating v gives the solution

$$y^{1-n} = Ce^{(1-n)\int f_1(x)dx} + (1-n)e^{(1-n)\int f_1(x)dx}\int f_n(x)e^{-(1-n)\int f_1(x)dx}\,dx, \quad C \in \mathbb{R}, \tag{6}$$

as given in Polyanin and Zaitsev (1995).

That this process could be formalized mathematically was established by the Norwegian mathematician Sophus Lie in a series of papers published in 1888-1893. The classical Lie point symmetry method (or group analysis) for finding exact solutions of differential equations was developed by Sophus Lie (1881) in the late nineteenth century and is based on the invariance of a differential equation under a continuous group of symmetries. For example, for a partial differential equation with one dependent and two independent variables, the method attempts to find a transformation

$$x_1 = x + \epsilon X(x,t,u) + O(\epsilon^2), \quad t_1 = t + \epsilon T(x,t,u) + O(\epsilon^2), \quad u_1 = u + \epsilon U(x,t,u) + O(\epsilon^2), \tag{7}$$

[where (7) is the *infinitesimal form* of the group] so that the original differential equation remains invariant. The transformations (7) are point symmetries as they map every solution of the differential equation to corresponding solutions of a related differential equation. In contrast, higher order (or Lie-Bäcklund) symmetries let the infinitesimal transformations depend on derivatives up to an arbitrary order (e.g. contact transformations depend on the first derivatives of the dependent variable with respect to the independent variables).

If a partial differential equation remains invariant under a point symmetry, it can often be transformed to reduce the number of variables by one. Consequently, an equation with one dependent and two independent variables would be reduced to an ordinary differential equation. Symmetries for a first order ordinary differential equation can be used to find canonical variables so that the equation can be reduced to an integrable form. If the ordinary differential equation is of higher order, canonical variables can be used to reduce the order of the equation by 1. There have been a number of references devoted to this topic in recent years including Olver (1986), Bluman and Kumei (1989) and Hill (1992).

Extensions to the classical point symmetry method include nonlocal (potential) symmetries, where the infinitesimals depend on integrals of the dependent variable (see, for example Bluman and Kumei (1989)), and nonclassical (or conditional) symmetries (Bluman and Cole (1974)) where the invariance of a system of partial differential equations comprising of the governing equation together with its invariant surface condition is sought.

Ibragimov (1994) shows that the Bernoulli equation (5) admits the symmetry

$$X(x,y) = \frac{e^{(1-n)\int f_1(x)dx}}{f_n(x)}, \quad Y(x,y) = f_1(x)\, y\, \frac{e^{(1-n)\int f_1(x)dx}}{f_n(x)}$$

with corresponding canonical coordinates

$$u = ye^{-\int f_1(x)dx}, \;\; v = \int f_n(x)e^{-(1-n)\int f_1(x)dx}\, dx.$$

The governing equation (5) reduces to the autonomous equation

$$\frac{du}{dv} = u^n.$$

Solving and writing in terms of the original variables recovers the solution (6).

A further instructive example relates to any equation of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{8}$$

which can be made separable by letting $v = \frac{y}{x}$. This is an implicit use of Lie group symmetry analysis as (8) admits the group of transformations

$$G : (x, y) \mapsto (\lambda x, \lambda y), \lambda > 0$$

as a symmetry group. Hence an invariant of (8) is $v = \frac{y}{x}$. This is an illustrative example used by Olver (1986).

Solutions of differential equations have been found through the implicit use of the symmetry properties of the governing equation without a full symmetry analysis. For example, Philip and Knight (1991) exploit similarity solutions of nonlinear flow equations without considering a symmetry analysis of the general class of equations. In other instances, exact solutions are only possible through the application of Lie group symmetry analysis. For example, Edwards and Broadbridge (1995) find solutions of Burgers' equation in two spatial dimensions using repeated symmetry reductions so that the partial differential equation with three independent variables is reduced to a second order ordinary differential equation which can subsequently be solved.

It may be possible that an equation does not possess Lie point symmetries and yet is easily integrable. Abraham-Shrauner et al. (1995) demonstrate a class of second-order equations which have no Lie point symmetries but admit nonlocal symmetries which subsequently allow a path for integration and hence general solutions to the original equation.

In summary, for the study of differential equations, the goal of symmetry analysis is to determine, for a given differential equation $\mathbf{L}u = 0$, the transformation group G (Olver (1986)) that leaves $\mathbf{L}u = 0$ invariant in the sense that the the transformed equation $\mathbf{L}_G u = 0$ has the same solutions as the original equation $\mathbf{L}u = 0$.

3 Application of symmetry analysis to von Bertalanffy

Consider the non-autonomous von Bertalanffy equation

$$\frac{dN}{dt} = \alpha(t)N^{\beta} - a(t)N^{b} + \psi(t), \tag{9}$$

where $\alpha(t), \beta, a(t)$ and b are all non-negative. We want to determine the transformation

$$\begin{aligned} t_1 &= t + \epsilon\tau(t,N) + O(\epsilon^2) \\ N_1 &= N + \epsilon\eta(t,N) + O(\epsilon^2) \end{aligned} \tag{10}$$

that leaves our governing equation (9) invariant. The symbol of the first prolongation of the group (10) is

$$\Gamma^{(1)} = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial N} + \eta_{[t]}\frac{\partial}{\partial N'} \tag{11}$$

where $\eta_{[t]} = D(\eta) - D(\tau)N'$ and $D = \frac{\partial}{\partial t} + N'\frac{\partial}{\partial N} + N''\frac{\partial}{\partial N'} + \cdots$. Hence

$$\eta_{[t]} = D(\eta) - D(\tau)N' = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial N}\frac{dN}{dt} - \left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial N}\frac{dN}{dt}\right)\frac{dN}{dt}.$$

Taking the first prolongation (11) of (9) and using the governing equation to eliminate the highest order derivative $\frac{dN}{dt}$ gives the single determining equation

$$\begin{aligned} &\left(-\frac{d\alpha}{dt}N^{\beta} + \frac{da}{dt}N^{b} - \frac{d\psi}{dt}\right)\tau(t,N) \\ &+ \left(-\frac{\beta\alpha(t)N^{\beta}}{N} + \frac{b\,a(t)\,N^{b}}{N}\right)\eta(t,N) + \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial N}\left(\alpha(t)N^{\beta} - a(t)N^{b} + \psi(t)\right) \\ &\qquad - \left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial N}\left(\alpha(t)N^{\beta} - a(t)N^{b} + \psi(t)\right)\right)\left(\alpha(t)N^{\beta} - a(t)\,N^{b} + \psi(t)\right) = 0. \end{aligned} \tag{12}$$

The determining equation (12) does not split into an overdetermined system, and so has an infinite set of solutions. If we were interested in a higher order equation, we would have an overdetermined system of determining equations to solve which would lead to a finite dimensional Lie symmetry algebra. As it is, we have a single determining equation to solve for $\tau(t,N)$ and $\eta(t,N)$. Often with a system of determining equations, it is possible to deduce a property of one of the dependent variables, for example, sometimes it may be possible to say that, in general, $\frac{\partial\tau}{\partial N} = 0$ and so there is an immediate simplification of the remaining determining equations. However, in this case, we will have to make some assumptions about the form of $\tau(t,N)$ and $\eta(t,N)$.

3.1 Case 1: τ and η separable

We assume that $\tau(t,N)$ and $\eta(t,N)$ are separable, that is, $\tau(t,N) = F_1(t)G_1(N)$ and $\eta(t,N) = F_2(t)G_2(N)$. The determining equation (12) becomes

$$\begin{aligned} &F_1(t)G_1(N)\left(-\frac{d\alpha}{dt}N^{\beta} + \frac{da}{dt}N^{b} - \frac{d\psi}{dt}\right) \\ &+ F_2(t)G_2(N)\left(-\alpha(t)N^{\beta-1}\beta + a(t)N^{b-1}b\right) + \frac{dF_2}{dt}G_2(N) \\ &+ F_2(t)\frac{dG_2}{dN}\left(\alpha(t)N^{\beta} - a(t)N^{b} + \psi(t)\right) \\ &+ \frac{dF_1}{dt}G_1(N)\left(-\alpha(t)N^{\beta} + a(t)N^{b} - \psi(t)\right) \\ &- F_1(t)\frac{dG_1}{dN}\left((\alpha(t))^2N^{2\,\beta} + (a(t))^2N^{2\,b} + (\psi(t))^2\right) \\ &+ 2\,F_1(t)\frac{dG_1}{dN}\left(\alpha(t)\,N^{\beta+b}a(t) - \alpha(t)N^{\beta}\psi(t) + a(t)N^{b}\psi(t)\right) = 0. \end{aligned} \tag{13}$$

To solve for $F_1(t)$ and $F_2(t)$, we could assume that each of the coefficients of expressions involving N are independent (assuming also that β, b are completely arbitrary). This leads to the set of equations

$$\begin{aligned}
G_1(N)N^{\beta}: &\quad F_1(t)\frac{d\alpha}{dt} + \frac{dF_1}{dt}\alpha(t) = 0 \\
G_1(N)N^{b}: &\quad F_1(t)\frac{da}{dt} + \frac{dF_1}{dt}a(t) = 0 \\
G_1(N): &\quad F_1(t)\frac{d\psi}{dt} + \frac{dF_1}{dt}\psi(t) = 0 \\
G_2(N)N^{\beta-1}: &\quad \beta F_2(t)\alpha(t) = 0 \\
G_2(N)N^{b-1}: &\quad bF_2(t)a(t) = 0 \\
G_2(N): &\quad \frac{dF_2}{dt} = 0 \\
\frac{dG_2}{dN}N^{\beta}: &\quad F_2(t)\alpha(t) = 0
\end{aligned}
\qquad
\begin{aligned}
\frac{dG_1}{dN}N^{2\beta}: &\quad F_1(t)\left((\alpha(t))^2 + (a(t))^2\right) = 0 \\
\frac{dG_1}{dN}N^{\beta+b}: &\quad 2F_1(t)\alpha(t)a(t) = 0 \\
\frac{dG_1}{dN}N^{\beta}: &\quad 2F_1(t)\alpha(t)\psi(t) = 0 \\
\frac{dG_1}{dN}N^{b}: &\quad 2F_1(t)a(t)\psi(t) = 0 \\
\frac{dG_1}{dN}: &\quad F_1(t)\,(\psi(t))^2 = 0 \\
\frac{dG_2}{dN}: &\quad F_2(t)\psi(t) = 0 \\
\frac{dG_2}{dN}N^{b}: &\quad F_2(t)a(t) = 0
\end{aligned}$$

The set of equations above immediately give $F_1(t) = F_2(t) = 0$ (if $\psi(t) \neq 0$), which just gives that $\tau(t, N) = \eta(t, N) = 0$. To find non-trivial solutions for τ and η, we could consider linear combinations of the terms from the LHS column.

3.2 Case 2: τ and η linear in N

Ibragimov (1994) considers the first order ODE

$$\frac{dy}{dx} = f(x, y) \tag{14}$$

and states that because there is not an overdetermined system of equations to solve we can *"hope to find solutions of certain forms only"*. He states that there is a sufficiently wide class of equations (14) that admit operators of the form

$$\Gamma \;=\; (a(x)y + b(x))\frac{\partial}{\partial x} + (c(x)y + d(x))\frac{\partial}{\partial y}.$$

We therefore assume for the determining equations for the von Bertalanffy equation (9) that τ and η are linear function of N, that is

$$\tau(t, N) = F_1(t)N + F_2(t), \qquad \eta(t, N) = G_1(t)N + G_2(t).$$

The determining equation (12) becomes

$$\begin{aligned}
&-\frac{d\alpha}{dt}N^{\beta}F_2(t) - \frac{dF_1}{dt}N^{1+\beta}\alpha(t) + \frac{dF_1}{dt}N^{1+b}a\,(t) - \frac{dF_1}{dt}N\psi(t) \\
&+ a(t)N^{b}bG_1\,(t) - \alpha(t)N^{\beta}\beta\,G_1\,(t) - 2\,F_1(t)\alpha(t)N^{\beta}\psi(t) + 2\,F_1(t)a\,(t)\,N^{b}\psi(t) \\
&- \frac{dF_2}{dt}\alpha(t)N^{\beta} + \frac{dF_2}{dt}a\,(t)\,N^{b} - F_1(t)\,(\alpha\,(t))^2\,N^{2\,\beta} - \frac{dF_2}{dt}\psi(t) \\
&- F_1\,(t)\,(\psi(t))^2 - \frac{d\psi}{dt}F_2(t) + 2\,F_1(t)\alpha(t)N^{\beta+b}a\,(t) - F_1(t)\,(a(t))^2\,N^{2\,b} \\
&+ \frac{da}{dt}N^{b}F_2(t) - \frac{d\psi}{dt}F_1(t)N - \frac{d\alpha}{dt}N^{1+\beta}F_1\,(t) + \frac{da}{dt}N^{1+b}F_1(t) \\
&- G_1(t)a\,(t)\,N^{b} + G_1(t)\alpha(t)N^{\beta} + \frac{dG_1}{dt}N + G_1(t)\psi(t) \\
&+ \frac{dG_2}{dt} + a(t)N^{b-1}bG_2(t) - \alpha(t)N^{\beta-1}\beta\,G_2(t) \;=\; 0.
\end{aligned} \tag{15}$$

Initially assuming that the coefficients of powers of N are independent (assuming that β and b are arbitrary) leads to the set of equations

$$\begin{aligned}
N^{\beta}: &\quad G_1(t)\alpha(t) - \frac{dF_2}{dt}\alpha(t) - \frac{d\alpha}{dt}F_2(t) - 2\,F_1(t)\alpha\,(t)\,\psi(t) - \alpha(t)\beta\,G_1(t) \;=\; 0\\
N^{b}: &\quad 2\,F_1(t)a(t)\psi(t) - G_1(t)a(t) + \frac{da}{dt}F_2(t) + \frac{dF_2}{dt}a(t) + a(t)bG_1(t) \;=\; 0\\
N^{\beta+1}: &\quad \frac{d\alpha}{dt}F_1(t) + \frac{dF_1}{dt}\alpha(t) \;=\; 0\\
N^{b+1}: &\quad \frac{dF_1}{dt}a\,(t) + \frac{da}{dt}F_1(t) \;=\; 0\\
N^{\beta+b}: &\quad 2\,F_1(t)\alpha(t)a\,(t) \;=\; 0\\
N^{2\beta}: &\quad F_1(t)\,(\alpha(t))^2 \;=\; 0\\
N^{2b}: &\quad F_1(t)\,(a(t))^2 \;=\; 0\\
N^{\beta-1}: &\quad \alpha(t)\beta\,G_2(t) \;=\; 0\\
N^{b-1}: &\quad a(t)bG_2(t) \;=\; 0\\
N^{1}: &\quad \frac{dG_1}{dt} - \frac{dF_1}{dt}\psi(t) - \frac{d\psi}{dt}F_1(t) \;=\; 0\\
N^{0}: &\quad \frac{dG_2}{dt} + G_1(t)\psi(t) - \frac{dF_2}{dt}\psi(t) - F_1(t)\,(\psi(t))^2 - \frac{d\psi}{dt}F_2(t) \;=\; 0
\end{aligned} \tag{16}$$

Solving the above equations immediately gives $F_1(t) = G_2(t) = 0$ (assuming that $\alpha(t), a(t), \beta, b \neq 0$). Consequently, from the coefficient of N^1, $G_1(t) = a_1,\ a_1 \in \mathbb{R}$. The remaining equations are

$$a_1\alpha(t)(1-\beta) - \frac{dF_2}{dt}\alpha(t) - \frac{d\alpha}{dt}F_2(t) \;=\; 0, \tag{17a}$$

$$a_1 a(t)(1-b) - \frac{dF_2}{dt}a(t) - \frac{da}{dt}F_2(t) \;=\; 0, \tag{17b}$$

$$a_1\psi(t) - \frac{dF_2}{dt}\psi(t) - \frac{d\psi}{dt}F_2(t) \;=\; 0. \tag{17c}$$

Equation (17c) can be solved (assuming $\psi(t) \neq 0$) to give

$$F_2(t) = \frac{a_1\int\psi(t)\,dt + c_1}{\psi(t)}, \qquad c_1 \in \mathbb{R}.$$

Equations (17a) and (17b) then give conditions

$$\alpha(t) = \frac{c_2\psi(t)}{(a_1\int\psi(t)dt + c_1)^{\beta}}, \qquad a(t) = \frac{c_3\psi(t)}{(a_1\int\psi(t)dt + c_1)^{b}}, \qquad c_2, c_3 \in \mathbb{R}, \tag{18}$$

that must be satisfied so that (9) admits the symmetry infinitesimals

$$\tau(t,N) = \frac{a_1\int\psi(t)dt + c_1}{\psi(t)}, \qquad \eta(t,N) = a_1 N. \tag{19}$$

The canonical coordinates corresponding to (19) are (letting $a_1 = 1$ and $c_1 = 0$ for simplicity)

$$u = \frac{N}{\int\psi(t)dt}, \qquad v = \ln\int\psi(t)dt,$$

which reduce the non-autonomous von Bertalanffy equation (9) to the autonomous equation

$$\frac{du}{dv} = c_2u^{\beta} - c_3u^{b} - u + 1.$$

Hence (9) has been reduced to an integrable form. Careful choices of β, b, c_2 and c_3 will lead to closed form solutions of the non-autonomous von Bertalanffy equation, although the case $\beta = 2$, $b = 1$ (or $\beta = 1$, $b = 2$) gives the Riccati equation, which has been discussed by Ibragimov (1994) and Gomez and Salas (2010).

This result has been found using the assumption that the various powers of N in (16) are independent. However, a number of cases could be considered: $\beta = b, \beta = b+1, \beta = 2b, \beta = b-1, \beta = 1, \beta = 0, b = 2\beta, b = 1, b = 0, \beta+1 = 2b, \beta+1 = b-1, \beta+1 = 0, b+1 = 2\beta, b+1 = \beta-1, b+1 = 0$. We note that the cases $b = 0$, $\beta = 0$ and $\beta = b$ could be discarded as (9) would collapse to have one non-linear term only. Although these cases may not lead to any additional symmetries, each case could be investigated to determine if there are special combinations of b and β with extra symmetries.

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