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A column generation approach for the scheduling of patrol boats to provide complete patrol coverage

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Abstract: The problem considered in this paper is motivated by fleet sizing questions faced by the Royal Australian Navy (RAN) and Customs and Border Protection (CBP). One possible requirement of a fleet is that it should be able to provide complete coverage of a set of stipulated patrol regions located around Australia's maritime approaches. This means that there must be at least one patrol boat on station in each patrol region at any given time. This is complicated by the fact that patrol boats cannot perform patrol operations indefinitely. Before a maximum operational time has expired, a patrol boat must return to a port for a mandatory resource replenishment break, such as crew layover time, maintenance or refuelling.

In the context of meeting the stated patrol capability, a natural question which arises is the following: What is the minimum number of patrol boats required to provide a set of patrol regions with complete coverage over a planning period? This problem must be addressed by taking into consideration the operational performance of the patrol boats, the geography of the patrol network (consisting of patrol regions and ports), and the duration of a mandatory resource replenishment break. A solution to the problem will consist of a schedule for each patrol boat, containing all the information on its activities throughout the planning period, including where and when it patrols (and for how long), and at which ports it replenishes resources (and when).

Our approach to the problem is based on column generation – an advanced optimisation technique employed within linear and integer programming to solve problems in which the number of variables (columns in the constraint matrix) is too large for the direct application of standard linear programming algorithms. The technique builds upon the revised simplex algorithm of linear programming by adding candidate variables to a restricted master problem. Variables are added sequentially to the restricted master problem by solving an optimisation subproblem (called the pricing subproblem) until no more variables are found to price out favourably. In its pure form, column generation addresses continuous variable problems, and just as the standard simplex algorithm is augmented with branch-and-bound to solve integer problems, column generation is augmented with branch-and-price.

In the context of scheduling patrol boats, a decision variable represents a feasible schedule for a patrol boat over the planning period. The restricted master problem is a set covering model, for which it is relatively straightforward to construct an initial feasible solution. The decision variables are generated via pricing subproblems – this involves solving shortest path problems over a custom designed resource-space-time network, which is constructed as a directed acyclic graph. The column generation algorithm is incorporated into a branch-and-price framework, where branching occurs on the arc variables of the underlying subproblem network. The branch-and-price technique is applied over a planning period of sufficient length so as to find cyclical patrol patterns for the boats.

We outline the column generation approach to the problem of routing and scheduling patrol boats with mandatory replenishment breaks and conduct a sensitivity analysis on an example patrol network. The results show how the column generation approach may assist decision makers by highlighting the tradeoffs between patrol boat numbers, endurance, replenishment break duration and achieving complete patrol coverage with sufficient schedule slack.

Keywords: *Branch-and-price, column generation, patrol boat scheduling, resource-space-time network*

1. INTRODUCTION

1.1. Problem Description

The problem relates to finding a set of patrol boat schedules which will ensure a continuous *patrol presence* in a predefined set of nominated maritime patrol regions. Given a set of patrol regions which are required to have at least one boat on patrol at any given time, the problem is to determine the minimum number of patrol boats required to meet the stated objective, subject to a number of operational and geographical constraints. This is a resource-constrained optimisation problem, as an individual patrol boat cannot perform patrol operations for an indefinite period of time. Patrol boats consume resources as they perform their duties, and therefore, each must return to a port on a regular basis in order to replenish depleted resources (e.g. refuelling, maintenance, or crew layover). The schedules for the boats must be feasible with respect to the geographical network (consisting of the patrol regions and ports), the capability of the patrol boats (that is, the maximum endurance), and the duration of the mandatory replenishment break (*layover time*), which occurs when a patrol boat returns to port. The diagram in Figure 1 is an example of a patrol operations network, consisting of four ports and seven patrol regions.

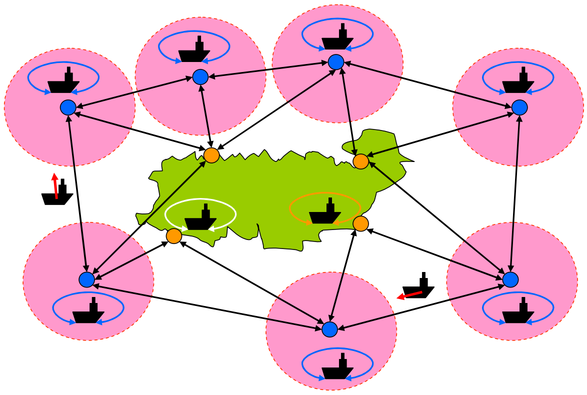

Figure 1. An illustration of the underlying patrol operations network and the possible states of the boats. The requirement is to have at least one boat on station in each patrol region at all times. The pink shaded circles are indicative of maritime patrol regions, but a region itself is represented by a blue vertex. Ports are displayed as orange vertices. An orange loop indicates that a boat is replenishing at a port (layover), a white loop indicates a boat at idle in port, and a blue loop implies that a boat is on active patrol. The black arcs represent feasible transit routes through the network and a red arrow is used to designate a boat in transit.

The problem, as stated above, is a combined routing and scheduling problem – not only do the routes of the patrol boats need to be determined, but also the timing of the activities that occur along the routes (e.g. for a given patrol boat, the duration of the patrol time at each patrol region it visits must be determined, as well as the timing of transitions between patrol regions and ports along its route). The problem, therefore, is to find the minimum number of patrol boats which can be routed and scheduled in a synchronous manner so as to ensure that there are no vacancies in patrol coverage. While superficially similar to the well-known vehicle routing problem with time windows (VRPTW), our problem is inherently different due to the requirement for continuous patrol presence. Henceforth, we refer to our problem as the *patrol boat scheduling problem with complete coverage* (PBSPCC).

1.2. Related Work

The problem, for the specific purposes outlined above, has not yet been examined in the academic literature. It is, however, related to a small number of problems from a collection of disparate domains, namely, patrol boat routing for fisheries surveillance, police car scheduling for accident hot spots, and scheduling unmanned aerial vehicles (UAVs) for long-term missions. We briefly outline the nature of these problems and the modelling and solution approaches that have been applied to them.

The work of Millar and Russell (2012) considered routing a fleet of patrol vessels over a geographical network of sea-based fishing grounds in order to maximise the total deterrence value of the surveillance effort. Patrol vessels in this surveillance model are routed through various pre-determined fishing grounds. A patrol vessel spends a pre-allocated amount of time at a fishing ground before moving to another fishing ground or back to port, subject to budget and time constraints. The scope of the problem is for short-term routing plans, since the surveillance prioritisation of fishing grounds occurs on a regular basis. The model is formulated via an extensive integer program, and since the size of the considered patrol vessel fleet is small and the planning horizon is short, it can be directly solved with state-of-the-art commercial solvers. There are some key differences between this formulation and the PBSPCC. These are the objectives, the predetermination of patrol time for each visit, the short planning horizon, and the omission of mandatory replenishment breaks for patrol vessels. It is also noteworthy that this problem routes patrol vessels independently, and hence, there is no need to synchronise the tasked activities.

The scheduling of police cars to provide maximum patrol coverage to a set of predefined accident hot spots with time windows has been recently studied by Keskin et al. (2012). Given a set of highway locations and time intervals at which traffic accidents have a high probability of occurring, the problem is to find patrol routes for a set of police cars so that the aggregate coverage of all accident hot spots is a maximum. This problem is modelled with respect to a mixed integer programming formulation and solved with recourse to heuristic techniques (local search and tabu search), due to the inability of state-of-the-art commercial solvers to find solutions. A continuous time modelling paradigm is used for the integer program and replenishment breaks for the police cars are not taken into consideration. Thus, while this problem is related to the PBSPCC in the aspect of providing patrol coverage to dispersed geographical locations, it does not consider replenishment or the requirement for complete patrol coverage. The problem is also not concerned with the question of long-term schedules.

The research conducted by Kim et al. (2013) considered the planning of mission trajectories for a system of resource capacitated UAVs. The problem is concerned with utilising a fleet of UAVs and a number of refuelling stations to provide "long-term mission fulfillment." A set of predefined space-time locations (referred to as split jobs) are required to be covered by exactly one UAV, subject to fuel capacity constraints. Like the approach taken by Keskin et al. (2012), the model is a mixed integer programming formulation, and is solved with a customised genetic algorithm. This problem is closely related to the PBSPCC in that it handles resource replenishment and the requirement for complete coverage to all split jobs. A fundamental difference, however, is the incorporation of predefined flight trajectories which the UAVs must follow.

A common characteristic of these studies is the way in which the respective problems are modelled. The position of each paper has been to write out an extensive integer programming formulation, which in some cases is nonlinear and requires linearisation in order to be amenable to commercial solvers. An additional drawback to writing out an all-encompassing integer program is the issue of computational intractability if replenishment breaks are considered. For example, the formulation used by Kim et al. (2013) introduces a large number of variables, such as X_{ijkr}, which has four indices and takes the value 1 if UAV k processes split job j after split job i during its r^{th} flight, and 0 otherwise. Our approach, which is based on column generation, differs from the standard approach taken in previous studies of related problems.

2. A COLUMN GENERATION APPROACH

Column generation (see Desrosiers and Lübbecke, 2005) is a technique for solving linear programming problems by dealing with the columns of the constraint matrix (or variables of the problem) in an implicit manner. The implicit handling of problem variables was first proffered by Ford and Fulkerson (1958), but was not put to practical use until Gilmore and Gomory (1961). The column generation process involves a coordinated cycling between two problems: a restricted master problem and a subproblem. The restricted master problem is a truncated rendering of a problem which involves too many variables to write down explicitly. By solving the restricted master problem as a linear program and obtaining its dual variables, a subproblem can be solved to determine a new column (variable) to add to the restricted master problem. The subproblem can accomplish this by casting the pricing step of the simplex algorithm (find a variable with negative reduced cost to enter the basis) as an optimisation problem. The process iterates between the restricted master problem and the subproblem, terminating when no further variables are determined to price out favourably. The column generation technique can be embedded within a branch-and-bound tree structure in order to solve large-scale integer programming problems (see Barnhart et al., 1998). Before outlining the construction of the restricted master problem and subproblem for the PBSPCC, we will introduce the overarching mathematical descriptors for the problem.

2.1. Problem Data

A patrol network is a directed graph $G = (V, A)$, where the set of vertices $V = \{1, \dots, n\}$ represents the number of distinct spatial regions and A is the set of directed arcs, that is, the set of feasible transitions (in space) between any two regions. The set of vertices is the union of two mutually exclusive sets, the set of ports $V_{\text{port}} = \{1, \dots, m\}$ and the set of patrol regions $V_{\text{patrol}} = \{m+1, \dots, n\}$. The number of ports is $m \geq 1$, where $m < n$, and the number of patrol regions is $n - m$. Associated with each arc $(i, j) \in A$, is a positive and integer-valued travel time, which we denote $t_{ij} \in \mathbb{Z}^+$. (Note: integer-valued travel times are not mandatory, so long as all travel times can be expressed as integer multiples of the time discretisation.)

The patrol boat maximum endurance is $T_E \in \mathbb{Z}^+$. This is the maximum duration of time that can be spent at economical cruising speed. It is assumed that a patrol boat travels at economical cruising speed while patrolling in a region and while transiting between any two spatial locations. A patrol boat must return to a port before the duration of its total travel time has reached the maximum endurance threshold. Once returning to port, a patrol boat remains there for a mandatory replenishment break (the layover time) given by $T_R \in \mathbb{Z}^+$. The planning period is $T \in \mathbb{Z}^+$, where $T \geq T_E + T_R$.

The requirement is that there be at least one patrol boat on station (that is, maintaining a patrolling presence) at all times in each patrol region. The planning period is divided into discrete intervals of time, according to the discretisation: $\Gamma = \{1,\ldots,T\}$. For a given patrol region $i \in V_{\text{patrol}}$, we require that there be a patrol boat on station for all time intervals $t \in \Gamma$. We define a *patrol period* to be a pairing of a discrete time interval and the index denoting a patrol region, that is, a patrol period is an element of the set:

$$\mathcal{L} = \{(i,t) \mid i \in V_{\text{patrol}},\ t \in \Gamma\} \tag{1}$$

If P is the set of all feasible patrol boat schedules, we define the binary parameter $a_{p\ell}$ to be equal to 1 if patrol period $\ell \in \mathcal{L}$ is patrolled in schedule $p \in P$, and 0 otherwise. In addition, we introduce the binary decision variable λ_p, which is equal to 1 if schedule $p \in P$ is selected in the solution, and 0 otherwise. Let $\overline{c}_p \in \mathbb{Z}^+$ be the time spent on patrol for schedule $p \in P$ and $c_p \in \mathbb{Z}^+$ be the time not spent on patrol, so that $T = \overline{c}_p + c_p,\ \forall p \in P$. We define $\overline{c}_{pi} \in \mathbb{Z}^+$ to be the time spent on patrol in region $i \in V_{\text{patrol}}$ for schedule $p \in P$. Finally, let $k_{\min}$ be a lower bound on the minimum number of schedules and let $\tau_{\min}$ be the theoretical minimum time a boat can spend in a state of non-patrol over the planning period T.

2.2. The Restricted Master Problem

The objective is to find the minimum number of patrol boats that will be able to supply every patrol region with complete coverage over the planning period. Since the number of feasible patrol boat schedules is too large to write down explicitly, the restricted master problem is constructed with a subset of the full set of feasible patrol boat schedules $P' \subset P$. The restricted master problem takes a set covering formulation with additional bounding constraints as follows (corresponding dual variables are indicated in parentheses):

$$\min \sum\nolimits_{p \in P'} \lambda_p, \quad \text{subject to:} \tag{2}$$

$$\sum\nolimits_{p \in P'} a_{p\ell} \lambda_p \geq 1,\ \ \forall \ell \in \mathcal{L} \qquad [\,\pi_\ell\,] \tag{3}$$

$$\sum\nolimits_{p \in P'} \overline{c}_p \lambda_p \geq |\mathcal{L}| \qquad [\,\alpha\,] \tag{4}$$

$$\sum\nolimits_{p \in P'} \overline{c}_{pi} \lambda_p \geq T, \quad \forall i \in V_{\text{patrol}} \qquad [\,\beta_i\,] \tag{5}$$

$$\sum\nolimits_{p \in P'} c_p \lambda_p \geq k_{\min} \tau_{\min} \qquad [\,\gamma\,] \tag{6}$$

$$\sum\nolimits_{p \in P'} \lambda_p \geq k_{\min} \qquad [\,\delta\,] \tag{7}$$

$$\lambda_p \geq 0,\ \ \forall p \in P' \tag{8}$$

The objective function (2) is to minimise the number of patrol boats used. The constraints given by (3) ensure that there is at least one patrol boat on station in each patrol region at all times. Constraints (4) through to (7) relate to various bounds that can be inferred from the patrol operations network and the capability of the patrol boats and are included to speed up the column generation process. In constraint (4), we have a bound on the minimum number of patrol hours required while (5) pertains to the bound on the minimum number of patrol hours required for individual patrol regions. Constraint (7) is a bound on the minimum number of patrol boats, and (6) places a bound on the minimum number of hours not spent on patrol. Finally, (8) stipulates non-negativity on the schedule decision variables. (The integrality conditions on the decision variables are relaxed, as column generation is concerned with solving the linear programming relaxation of the integer problem.) The dual variables for the constraints are determined when the linear program (2) – (8) is solved, and these are used to feed information to the subproblem in order to find new columns / variables to add to the restricted master problem.

2.3. The Subproblem

The subproblem construction that we have developed for the PBSPCC is a directed acyclic graph which implicitly accounts for resource consumption and replenishment along a schedule's trajectory through space and time. The subproblem network accounts for patrol boat transitions in space and time and ensures that boats return to a port for replenishment before their resource levels are depleted. An important feature of the network's design is that the subproblem can be solved with a pure shortest path algorithm (edge relaxation on a topologically sorted vertex list) due to the way in which resource consumption and replenishment are encoded. Our design was chosen for ease of extensibility and to handle multiple replenishment breaks.

The underlying network is a *resource-space-time* (RST) graph, which we define to be a directed acyclic graph $H = (U, B)$, where U is the set of vertices, B is the set of arcs and each vertex in this graph has an associated triple (v, d, t). Here, $v \in V$ is an index of a patrol region or port, d is an integer-valued measure of the resource level of a patrol boat at that location, and $t \in \Gamma \cup \{0\}$ is the time instance. In this sense, each spatial location is represented as a layered series of vertices. The vertical layers are indicative of how much resource has been consumed at that location, and the horizontal position is indicative of the time interval. Under this paradigm, ports are handled differently to patrol regions. Ports have two layers of vertices (one for the un-replenished state and one for the replenished state), which are connected by directed arcs representing the duration of the replenishment break.

As an illustration of this concept, consider the diagram in Figure 2. This is an RST graph for a patrol network with one port (blue vertices) and two patrol regions (pink vertices for region 01, and green vertices for region 02). The horizontal spacing between vertices is indicative of the time discretisation, while the vertical layers represent the possible resource levels for a given location and time period. Blue arcs are for replenishment at port, whereas orange arcs describe a patrol boat sitting idle in port. The black arcs are used to describe a boat on patrol in a patrol region, and all other arcs represent feasible transitions between the various spatial locations. The patrol periods that need to be covered are demarcated by the dashed vertical lines.

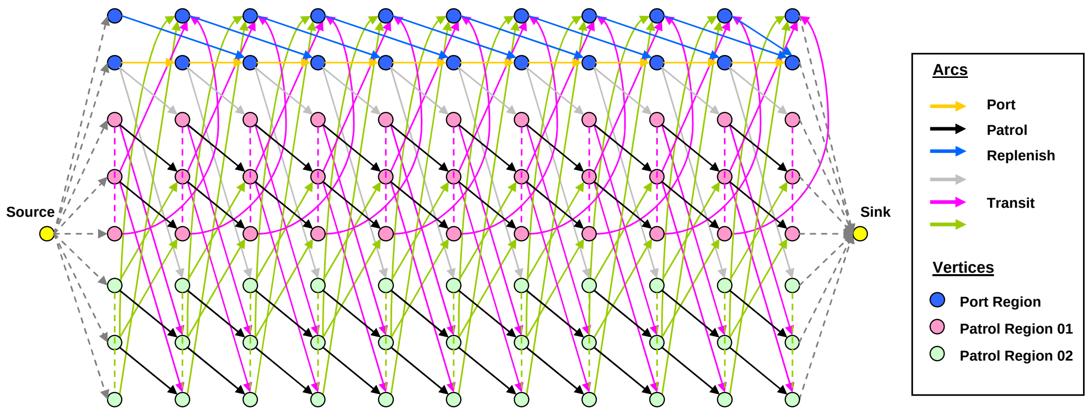

Figure 2. An illustrative example of the underlying RST subproblem network.

For the RST network, we define binary decision variables x_{ij} to be 1 if arc $(i, j) \in B$ is used, and 0 otherwise. We define $B_{\text{patrol}} \subset B$ as the set of all patrol arcs. The function $\phi : B \to \mathcal{L}$ maps patrol arcs to their respective patrol period index and the function $\psi : B \to V$ maps patrol arcs to their patrol region index. Each arc flow variable has an associated cost (c_{ij}) and travel time (t_{ij}). The costs on the arc variables, which may be negative, are expressed in terms of the restricted master problem dual variables as follows:

$$c_{ij} = \begin{cases} -\pi_{\phi(i,j)} - \left(\alpha + \beta_{\psi(i,j)}\right) t_{ij}, & (i,j) \in B_{\text{patrol}} \\ -\gamma t_{ij}, & \text{otherwise} \end{cases} \tag{9}$$

At each iteration of the column generation algorithm, the subproblem determines whether there is a path p through the RST network with minimum negative reduced cost, as given by (10). If a negative reduced cost path exists, it is added as a new column in the restricted master problem.

$$\overline{r}^* = \arg\min\left\{\sum_{(i,j)\in p} c_{ij} x_{ij} + (1 - \delta) \mid p \text{ is a feasible path}\right\} \tag{10}$$

2.4. Branch-and-Price

The column generation procedure iterates between the restricted master problem and the subproblem until no more columns can be found with negative reduced cost. When termination occurs, we have a solution to the linear relaxation of the integer problem. This is called column generation at the root node. By examining the flow of the variables on the underlying subproblem network, we can embed the column generation process into a branch-and-bound tree (this is called *branch-and-price*). It is important to note that the branching is on the subproblem arc flow variables, not on the restricted master problem variables. This preserves the integrity of the subproblem's structure, allowing it to remain unaltered as we move throughout the branch-and-bound tree. The branching rule we have adopted is depth-first search, with branching terminating when an optimal integer solution has been found by pruning the tree with the bounds produced by column generation.

3. COMPUTATIONAL EXPERIMENT (SENSITIVITY ANALYSIS)

The column generation approach has been tested by applying it to an example patrol network, characterised by three ports and seven patrol regions (Figure 3). This example network may be indicative of ports based on a central island with the requirement to supply complete coverage to the surrounding designated patrol regions. The maximum transit time between any two locations is 3 time units (e.g. moving from patrol region 05 to patrol region 08), and the smallest is 1 time unit (e.g. moving from patrol region 05 to patrol region 06).

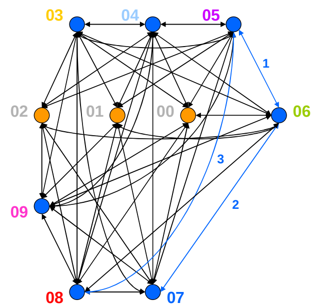

Figure 3. An example network consisting of three ports and seven patrol regions, with indicative travel times shown.

The column generation algorithm, via the pricing subproblem, seeks to find schedules (for the minimum number of patrol boats) which behave as cyclic permutations over a period of T or $2T$, where $T = T_E + T_R$. Cyclical schedules provide a straightforward way of ensuring that a schedule can be extended indefinitely into the long-term future, providing each patrol boat with an ongoing and regular set of activities. This is made possible by the layers in the RST network design and is achieved through a customised heuristic, which cuts off integer solutions which cannot be extended into the next planning period. An outline of this heuristic is beyond the scope of this paper and its exposition will be a matter for a future publication, along with more extensive computational testing. An illustration of a cyclical scheduling solution is shown in Figure 4.

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(1) |06|05|05|05|05|05|05|05|05|05|05|05|05|05|05|05|05|05|05|05|00|06|06|06|06|06|06|06
(2) |08|08|08|08|08|08|08|..|01|..|07|07|07|07|07|07|07|07|07|07|07|07|07|07|07|08|08|08
(3) |05|05|04|04|04|04|04|04|04|04|04|04|04|04|04|04|04|04|00|05|05|05|05|05|05|05|05|05
(4) |04|04|04|03|03|03|03|03|03|03|03|03|03|03|03|02|09|09|09|09|09|09|09|09|09|09|09|09|09
(5) |09|09|09|09|09|09|09|09|09|09|09|09|02|03|03|03|03|03|03|03|03|03|03|03|03|03|03|03
(6) |03|03|03|03|02|..|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|08|..|00
(7) |06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|06|00|..|07|07|07|07
(8) |07|07|07|07|07|07|07|07|07|07|07|09|09|09|09|09|01|04|04|04|04|04|04|04|04|04|04|04
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Figure 4. An optimal scheduling solution for the example network, where the maximum endurance (T_E) is 28 time units and the replenishment break (T_R) is 0 time units. The solution is a cyclic permutation of schedules over the planning period (T), with the identity mapping obtained on $5T$. Schedules (1) – (3) each map to themselves. Schedule (4) is extended by appending schedule (5), which is continued by (6), then subsequently by (7) and (8), upon which we are returned to schedule (4).

The computational tests were performed by varying the maximum endurance of the patrol boats between 10 time units and 35 time units and varying the length of the mandatory replenishment period at port from 0 time units to 5 time units. The time to solve on a 3.16 GHz dual-core desktop using CPLEX 12.2 ranges from a few seconds to a few minutes, depending on the length of the planning period. For each pairing of maximum endurance and replenishment break, the minimum number of patrol boats was recorded and the amount of patrol coverage supplied (100% being the minimum requirement). The behaviour of the minimum number of patrol boats required and the patrol coverage effort can be seen in Figure 5a and Figure 5b, respectively. (All results are given with respect to cyclical schedules over a period of T or $2T$.)

Figure 5a shows how the number of patrol boats changes with maximum endurance and replenishment time. Unsurprisingly, the numbers decrease as endurance increases and replenishment time decreases. This type of chart can be used by decision makers to set performance goals for patrol boats based on desired numbers. Furthermore, decision makers can derive insight into the fragility of the patrol boats' capability to meet the complete coverage requirement by looking at the level of patrol coverage (schedule slack) afforded by each combination of maximum endurance and replenishment time. Figure 5b illustrates this, showing the excess patrol coverage for replenishment times of 2 and 4 time units. We note the steep drop in excess patrol coverage whenever the number of boats decreases by one. Charts of this type, when used in conjunction with a patrol boat numbers chart, may inform decision makers of the marginal return gained by increasing the maximum endurance. For a set of schedules P over T, the excess patrol coverage is:

$$\%\ \text{Patrol Coverage} = 100 \times \left(T \times |V_{\text{patrol}}|\right)^{-1} \sum\nolimits_{p \in P} \bar{c}_p \lambda_p \tag{11}$$

4. CONCLUSIONS

We have demonstrated how a column generation approach may be applied to the problem of scheduling and routing a fleet of patrol boats over a patrol network to provide complete coverage on an ongoing basis. The model is expressed succinctly through a restricted master problem in which patrol boat schedules are represented implicitly. This is a departure from extant approaches to related problems which have been modelled as pure integer programs. Such formulations have a greater tendency to become computationally intractable and therefore must be solved heuristically. The subproblem utilises a modelling paradigm in which resource consumption and replenishment are encoded into the network construction. This enables the subproblem to be solved with a pure shortest path algorithm. Computational tests on an example network show that this approach can be used to find cyclical schedules. The technique can be used to compare various options for patrol boat fleet sizing in terms of maximum endurance, replenishment break selection and the amount of schedule slack. Future work will examine the applicability of the column generation technique to a variety of patrol networks.

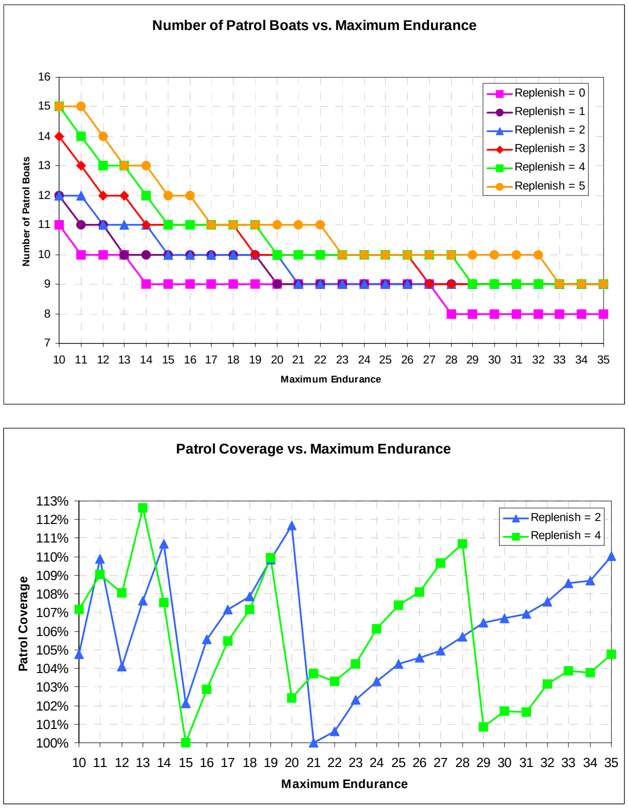

Figure 5. (a) Sensitivity analysis – number of patrol boats, (b) Sensitivity analysis – patrol coverage.

REFERENCES

Barnhart, C., Johnson, E.L., Nemhauser, G.L., Savelsbergh, M.W.P., and Vance, P.H. (1998). Branch-and-price: Column generation for solving huge integer programs. *Operations Research*, 46, 316–329.

Desrosiers, J., and Lübbecke, M.E. (2005). Selected topics in column generation. *Operations Research*, 53, 1007–1023.

Ford, L.R., and Fulkerson, D.R. (1958). A suggested computation for maximal multicommodity network flows. *Management Science*, 5, 97–101.

Gilmore, P.C., and Gomory, R.E. (1961). A linear programming approach to the cutting-stock problem. *Operations Research*, 9, 849–859.

Keskin, B.B., Li, S.R., Steil, D., and Spiller, S. (2012). Analysis of an integrated maximum covering and patrol routing problem. *Transportation Research Part E: Logistics and Transportation Review*, 48, 215–232.

Kim, J., Song, B.D., and Morrison, J.R. (2013). On the scheduling of systems of UAVs and fuel service stations for long-term mission fulfillment. *Journal of Intelligent & Robotic Systems*, 70, 347–359.

Millar, H.H., and Russell, S.N. (2012). A model for fisheries patrol dispatch in the Canadian Atlantic offshore fishery. *Ocean & Coastal Management*, 60, 48–55.