Efficient rare event simulation for random graphs via importance sampling

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Abstract: A Gilbert graph, also known as a random geometric graph, is a mathematical model used to study the properties of networks formed by connecting points in a space according to some interaction rule. This work introduces a novel importance-sampling-based estimator for rare events of Gilbert Graphs. In particular, a Gilbert graph on a window $W \subset \mathbb{R}^d$ is generated by a Poisson point process, where we draw an edge between any two points if the distance between them is at most r, which is referred to as the *interaction range*. Such graphs are useful for modeling communication, transportation, and social networks, as well as in computer science for studying the properties of algorithms, such as distributed algorithms and routing algorithms.

Estimating the rare events in Gilbert graphs is a challenging task. For instance, we may ask: "What is the probability that the Gilbert graph has at most half its expected number of edges?" This probability can be extremely small when the window W is large and/or the intensity of the point process is high. In such cases, a naive Monte Carlo method, which simply generates several independent samples of the Poisson process and provides an estimate of the rare event as the fraction of times the event is satisfied, is not efficient precisely because the target event is rare.

To address this issue, recently Hirsch et al. (2020) developed a conditional Monte Carlo estimator for the above edge-count-related rare event. This approach is better than the naive method in terms of relative variance for the same computational budget. However, as we see in the talk, the proposed novel importance-sampling method substantially outperforms both the naive and conditional Monte Carlo methods. This importance-sampling-based method is based on the grid-based approach proposed in Moka et al. (2017). Our new method is general in the sense that it can be used beyond edge count, including rare events associated with degrees of the graph, connected components in the graph, and number of triangles in the graph.

The key idea is as follows: Suppose A denotes the occurrence of the rare event of interest for a Gilbert graph. Then, the naive estimator Z_{nmc} of $\mathbb{P}(A)$ is a a sample mean of independent and indentically distributed (iid) samples of $Y = I(A)$. Now suppose M denotes the number of iid points generated on W before A is not satisfied. The conditional Monte Cralo estimator Z_{cmc} is a sample mean of iid samples of $\bar{Y} = F(M)$, where F is the cummulative disctribution function of the underlying Poisson counting variable with rate $\lambda = \lambda_0 \mathsf{Vol}(W)$ with λ_0 denoting the intensity of the Poisson process and Vol is the volume of W. That is,

$$\bar{Y} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) I(A \text{ is satisfied with } n \text{ points}).$$

By replacing the binary random variables $I(A \text{ is satisfied})$ with random variables L_n of the same mean but with a lesser variance via importance sampling, we obtain an importance-sampling-based estimator Z_{ismc} as a sample mean of iid samples of

$$\widetilde{Y} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) L_n.$$

References

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